Module–V:
Complex Variable-Integration

13 CHAPTER

Complex Integration
(Contour integrals, Cauchy-Goursat theorem, Cauchy integral Formula)

13.1 INTRODUCTION (LINE INTEGRAL)

In case of real variable, the path of integration of $\int_a^b f(x)\,dx$ is always along the x-axis from $x = a$ to $x = b$. But in case of a complex function $f(z)$ the path of the definite integral $\int_a^b f(z)dz$ can be along any curve from $z = a$ to $z = b$.

$z = x + iy \Rightarrow dz = dx + idy$ ... (1) $dz = dx$ if $y = 0$ ... (2) $dz = idy$ if $x = 0$ ... (3)

In (1), (2), (3) the directions of dz are different. Its value depends upon the path (curve) of integration. But the value of integral from a to b remains the same along any regular curve from a to b.

In case the initial point and final point coincide so that c is a closed curve, then this integral is called *contour integral* and is denoted by $\oint_C f(z)\,dz$.

If $f(z) = u(x, y) + iv(x, y)$, then since $dz = dx + idy$,

we have

$$\oint_C f(z)\,dz = \int_C (u + iv)(dx + idy)$$

$$= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$

which shows that the evaluation of the line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

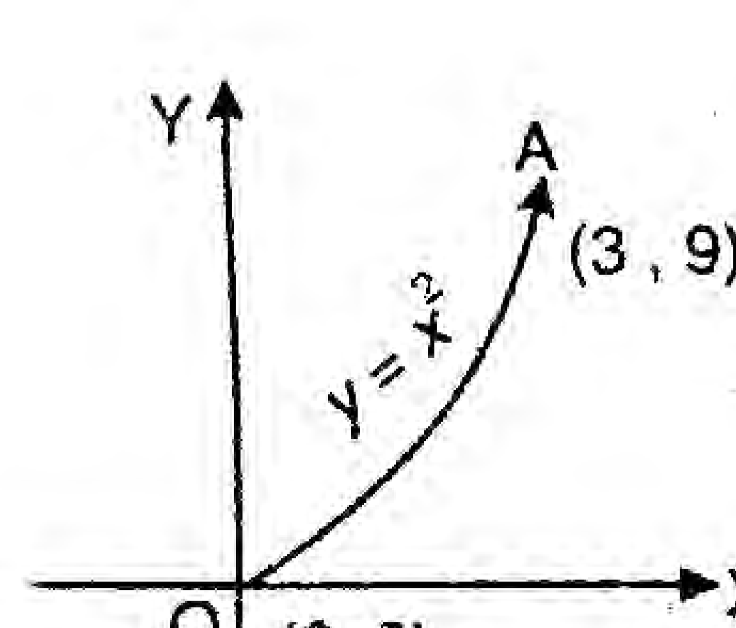
Let us consider a few examples:

Real integral

Example 1. *Find the value of the integral* $\int_C (x + y)dx + x^2y\,dy$

(a) along $y = x^2$, *having* (0, 0), (3, 9) *end points.*
(b) along $y = 3x$ *between the same points.*
Do the values depend upon path ?

Solution. $\int_C (x + y)\,dx + x^2y\,dy$ ...(1)



Let $P = x + y,\ Q = x^2y$

$$\frac{\partial P}{\partial y} = 1, \quad \frac{\partial Q}{\partial x} = 2xy \quad \text{or} \quad \frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

The integrals are not independent of path.

15. The value of the integral $\int_{(3,0)}^{(4,2)} (2y^2 + x)\, dx + (3y - x)\, dy$ along the figure $x^2 = t^2 + 3,\ y = 2t$ is

(*a*) 0 (*b*) 1 (*c*) 4/3 (*d*) 41/6 **Ans.** (*d*)

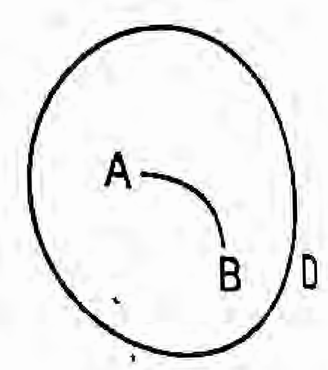
16. along $\int_0^{1+i} (x^2 - iy)\, dz$ the path $y = x$ is equal to

(*a*) $-\frac{1}{3}(2+i)$ (*b*) $\frac{1}{3}(2+i)$ (*c*) $\frac{1}{6}(2+i)$ (*d*) $\frac{1}{6}(5-i)$ **Ans.** (*d*)

17. The value of the line integral $\int_C (y^2\, dx + x^2\, dy)$ where C is of the square $-1 \le x \le 1, -1 \le y \le 1$, is

(*a*) 0 (*b*) 2 (x + y) (*c*) 4 (*d*) 4/3 **Ans.** (*a*)

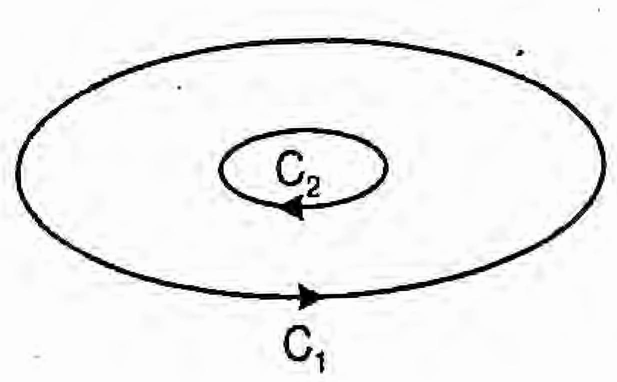
13.2 IMPORTANT DEFINITIONS



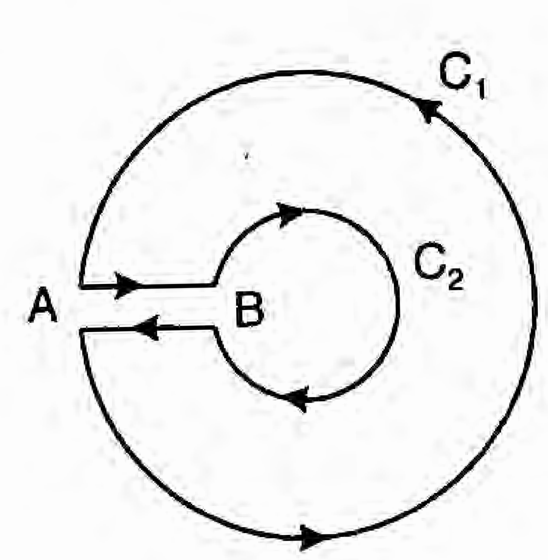
(*i*) **Simply connected Region.** A connected region is said to be a simply connected if all the interior points of a closed curve C drawn in the region D are the points of the region D.

(*ii*) **Multi-Connected Region.** Multi-connected region is bounded by more than one curve. We can convert a multi-connected region into a simply connected one, by giving it one or more cuts.

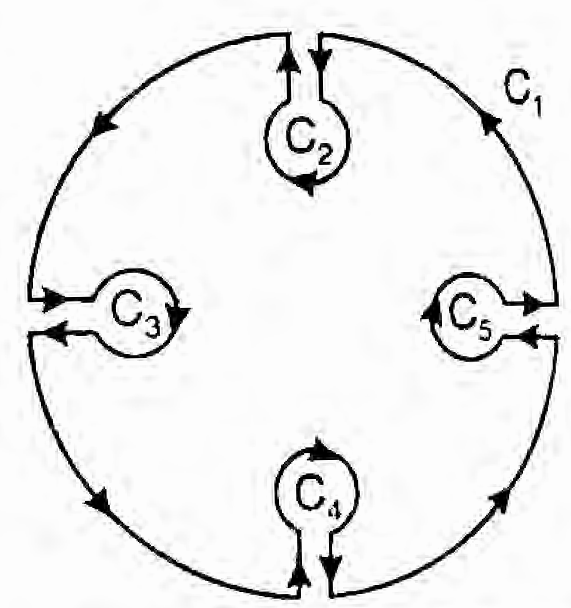
Note. A function $f(z)$ is said to be **meromorphic** in a region R if it is analytic in the region R except at a finite number of poles.



Multi-Connected Region



Simply Connected Region



Simply Connected Region

(*iii*) **Single-valued and Multi-valued function**

If a function has only one value for a given value of z, then it is a single-valued function.

For example $f(z) = z^2$

If a function has more than one value, it is known as multi-valued function.

For example $f(z) = z^{\frac{1}{2}}$

(*iv*) **Limit of a function**

A function $f(z)$ is said to have a limit l at a point $z = z_0$, if for an arbitrarily chosen positive number ε, there exists a positive number δ, such that

$$|f(z) - l| < \varepsilon \text{ for } |z - z_0| < \delta$$

It may be written as $\lim_{z \to z_0} f(z) = l$

(*v*) **Continuity**

A function $f(z)$ is said to be continuous at a point $z = z_0$ if for a given arbitrary positive number ε, there exists a positive number δ, such that

$$|f(z) - f(z_0)| < \varepsilon \text{ for } |z - z_0| < \delta$$

In other words, a function $f(z)$ is continuous at a point $z = z_0$ if

(*a*) $f(z_0)$ exists (*b*) $\lim_{z \to z_0} f(z) = f(z)_{z=0}$

(*vi*) **Multiple point.** If an equation is satisfied by more than one value of the variable in the given range, then the point is called a multiple point of the arc.

(*vii*) **Jordan arc.** A continuous arc without multiple points is called a Jordan arc.

Regular arc. If the derivatives of the given function are also continuous in the given range, then the arc is called a regular arc.

(*viii*) **Contour.** A contour is a Jordan curve consisting of continuous chain of a finite number of regular arcs.

The contour is said to be closed if the starting point A of the arc coincides with the end point B of the last arc.

(*ix*) **Zeros of an Analytic function.**

The value of z for which the analytic function $f(z)$ becomes zero is said to be the zero of $f(z)$. **For example,** Zeros of $z^2 - 3z + 2$ are $z = 1$ and $z = 2$,

(2) Zeros of cos z is $\pm(2n-1)\frac{\pi}{2}$, where $n = 1, 2, 3$.......... .

13.3 CAUCHY'S INTEGRAL THEOREM (*Cauchy-Gaursat Theorem*)

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.

Proof. Let the region enclosed by the curve c be R and let

$$f(z) = u + iv,\quad z = x + iy,\quad dz = dx + idy$$

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

$$= \iint_S \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \quad \text{(By Green's theorem)}$$

Replacing $-\frac{\partial v}{\partial x}$ by $\frac{\partial u}{\partial y}$ and $\frac{\partial v}{\partial y}$ by $\frac{\partial u}{\partial x}$ we get

$$\int_c f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_c \left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial x}\right) dx\,dy = 0 + i0 = 0$$

Proved.

$$\Rightarrow \quad \int_c f(z)\,dz = 0$$

Note. If there is no pole inside and on the contour then the value of the integral of the function is zero.

Example 16. *Verify Cauchy's Theorem for the function $f(z) = e^{iz}$ along the boundary of the triangle with vertices at the points $1 + i$, $-1 + i$ and $-1 - i$.*

(G.B.T.U., III Sem., Dec. 2012 April 2012)

Solution. Integration of e^{iz} along the boundary of Δ ABC = Integration of e^{iz} along AB, BC and CA

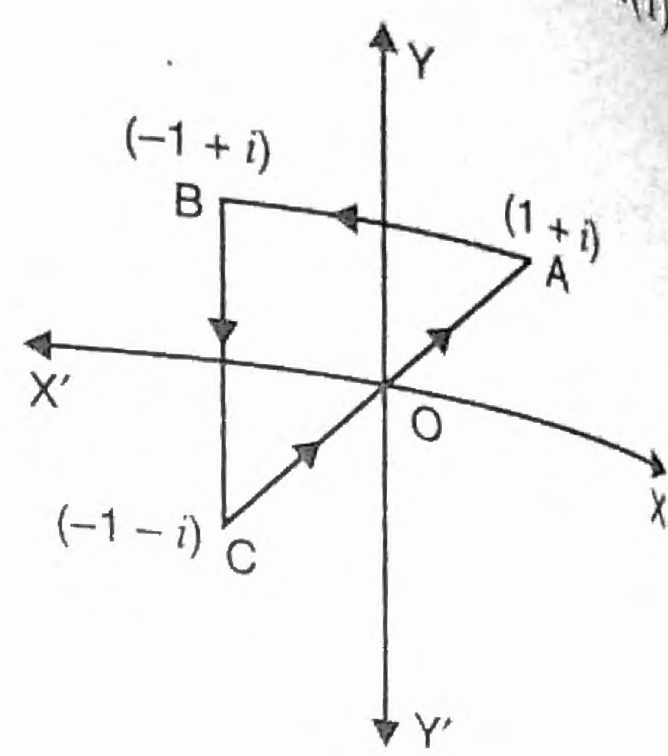
$$= \int_{AB} e^{iz}\,dz + \int_{BC} e^{iz}\,dz + \int_{CA} e^{iz}\,dz$$

$$= I_1 + I_2 + I_3$$

Now, I_1 = Integration of e^{iz} along AB ...(1)

$$= \int_{1+i}^{-1+i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{1+i}^{-1+i}$$

$$= \frac{1}{i}\left[e^{i(-1+i)} - e^{i(1+i)}\right] = \frac{1}{i}\left[e^{-i-1} - e^{i-1}\right]$$



I_2 = Integration of e^{iz} along BC

$$= \int_{-1+i}^{-1-i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{-1+i}^{-1-i}$$

$$= \frac{1}{i}\left[e^{i(-1-i)} - e^{i(-1+i)}\right] = \frac{1}{i}\left[e^{-i+1} - e^{-i-1}\right]$$

I_3 = Integration of e^{iz} along CA

$$= \int_{-1-i}^{1+i} e^{iz}\,dz = \left[\frac{e^{iz}}{i}\right]_{-1-i}^{1+i} = \frac{1}{i}\left[e^{i(1+i)} - e^{i(-1-i)}\right] = \frac{1}{i}(e^{i-1} - e^{-i+1})$$

Now $\int_{ABC} e^{iz}\,dz = I_1 + I_2 + I_3$

$$= \frac{1}{i}(e^{-i-1} - e^{i-1}) + \frac{1}{i}(e^{-i+1} - e^{-i-1}) + \frac{1}{i}(e^{i-1} - e^{-i+1})$$

$$= \frac{1}{i}[e^{-i-1} - e^{-i-1} + e^{-i+1} - e^{-i-1} + e^{-i-1} - e^{-i+1}] = 0 \quad ...(2)$$

According to Cauchy Theorem,

If a function $f(z)$ is analytic and its derivative $f'(z)$ continuous at all points inside and on a simple closed curve c, then $\int_c f(z)\,dz = 0$ ...(3)

From (2) and (3), Cauchy Theorem is verified. **Verified**

Example 17. *Verify Cauchy's theorem for the function $f(z) = 3z^2 + iz - 4$ along the perimeter of square with vertices $1 \pm i$, $-1 \pm i$.* (*U.P., III Semester June 2011*)

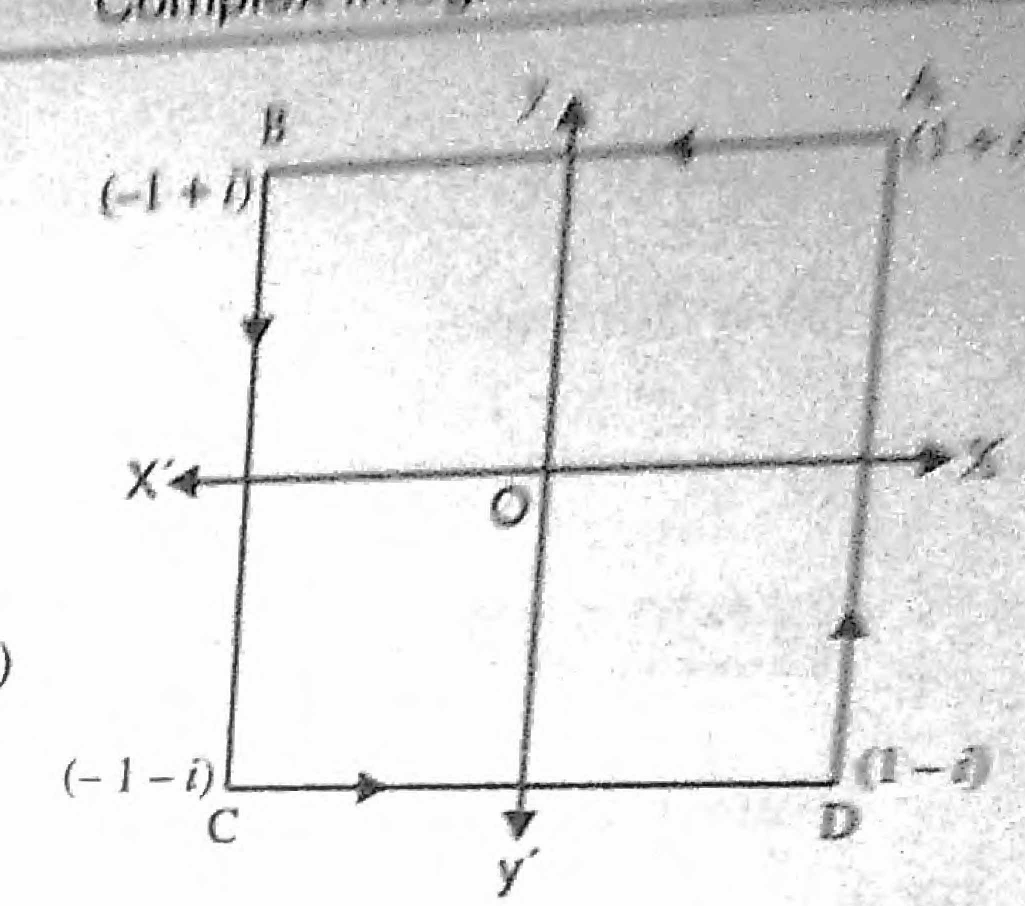
Solution. Integration of $3z^2 + iz - 4$ along the boundary of the square ABCD = Integration of $3z^2 + iz - 4$ along AB, BC, CD, and DA.

$$= \int_{AB}(3z^2 + iz - 4)\,dz + \int_{BC}(3z^2 + iz - 4)\,dz + \int_{CD}(3z^2 + iz - 4)\,dz + \int_{DA}(3z^2 + iz - 4)\,dz$$

$$= I_1 + I_2 + I_3 + I_4$$

Now $I_1 = \int_{AB}(3z^2 + iz - 4)\,dz = \int_{1+i}^{-1+i}(3z^2 + iz - 4)\,dz$

$$= \left[z^3 + \frac{i}{2}z^2 - 4z\right]_{1+i}^{-1+i}$$



$$= \left[(-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i)\right] - \left[(1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i)\right] \quad ...(1)$$

$$I_2 = \int_{BC}(3z^2 + iz - 4)\,dz = \int_{-1+i}^{-1-i}(3z^2 + iz - 4)\,dz$$

$$= \left[z^3 + \frac{iz^2}{2} - 4z\right]_{-1+i}^{-1-i}$$

$$= \left[(-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i)\right] - \left[(-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i)\right] \quad ...(2)$$

$$I_3 = \int_{CD}(3z^2 + iz - 4)\,dz = \int_{-1-i}^{1-i}(3z^2 + iz - 4)\,dz = \left[z^3 + \frac{i}{2}z^2 - 4z\right]_{-1-i}^{1-i}$$

$$= \left[(1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i)\right] - \left[(-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i)\right] \quad ...(3)$$

$$I_4 = \int_{DA}(3z^2 + iz - 4)\,dz = \int_{1-i}^{1+i}(3z^2 + iz - 4)\,dz = \left[z^3 + \frac{i}{2}z^2 - 4z\right]_{1-i}^{1+i}$$

$$= \left[(1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i)\right] - \left[(1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i)\right] \quad ...(4)$$

Adding (1), (2), (3), and (4), we get

$$I_1 + I_2 + I_3 + I_4 = (-1+i)^3 + \frac{i}{2}(-1+i)^2 - 4(-1+i) - (1+i)^3 - \frac{i}{2}(1+i)^2 + 4(1+i)$$
$$+ (-1-i)^3 + \frac{i}{2}(-1-i)^2 - 4(-1-i) - (-1+i)^3 - \frac{i}{2}(-1+i)^2 + 4(-1+i)$$
$$+ (1-i)^3 + \frac{i}{2}(1-i)^2 - 4(1-i) - (-1-i)^3 - \frac{i}{2}(-1-i)^2 + 4(-1-i)$$
$$+ (1+i)^3 + \frac{i}{2}(1+i)^2 - 4(1+i) - (1-i)^3 - \frac{i}{2}(1-i)^2 + 4(1-i)$$
$$= 0 \quad ...(5)$$

In the square $ABCD$ there is no pole, so by Cauchy Goursat theorem,

$$\int_{ABCD}(3z^2 + iz - 4)\,dz = 0 \quad ...(6)$$

From (5) and (6), Cauchy Goursat theorem is verified.

Example 18. *Find the integral* $\int_c \frac{3z^2 + 7z + 1}{z + 1}\,dz$, where C is the circle $|z| = \frac{1}{2}$.

Solution. Poles of the integrand are given by putting the denominator equal to zero.

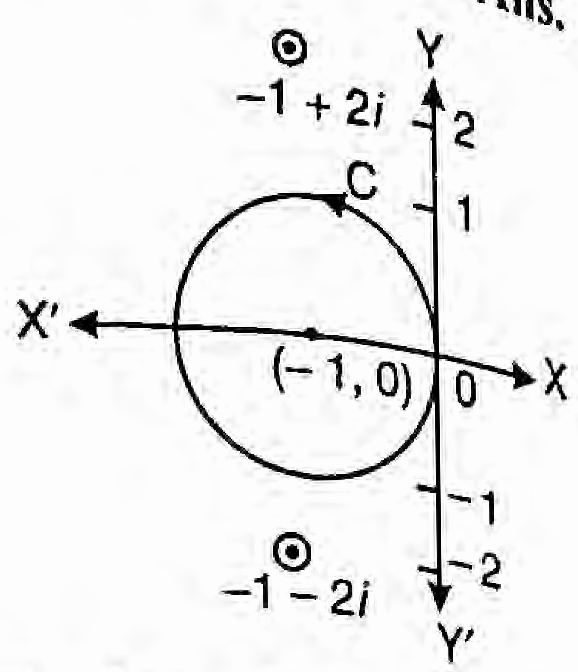
$z + 1 = 0 \Rightarrow z = -1$

The given circle $|z| = \frac{1}{2}$ with centre at $z = 0$ and radius $\frac{1}{2}$ does not enclose any singularity of the given function.

$$\int_C \frac{3z^2+7z+1}{z+1}\,dz = 0 \quad \text{(By Cauchy Goursat Theorem)}$$

Ans.

Example 19. *Find the value of* $\int_C \frac{z+4}{z^2+2z+5}\,dz$, *if C is the circle* $|z+1| = 1$.



Solution. Poles of integrand are given by putting the denominator equal to zero.

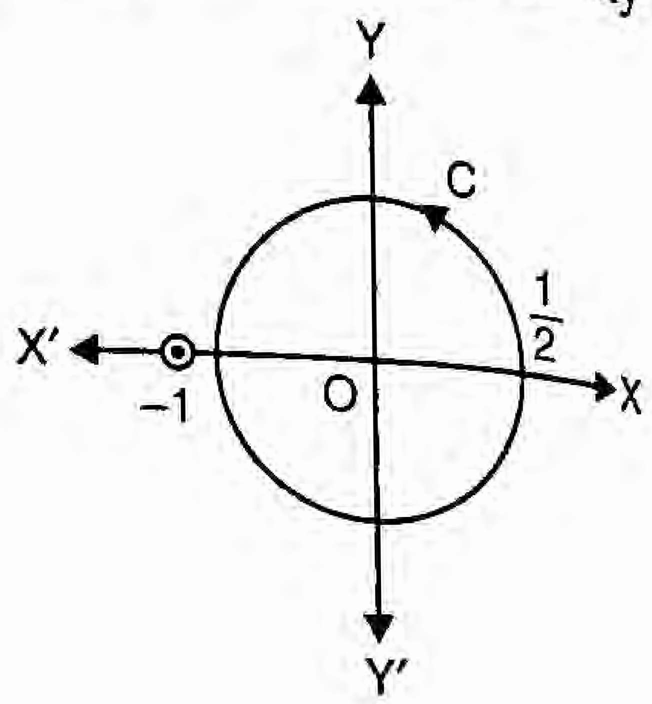
$$z^2 + 2z + 5 = 0$$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

The given circle $|z+1| = 1$ with centre at $z = -1$ and radius unity does not enclose any singularity of the function $\frac{z+4}{z^2+2z+5}$.

$$\therefore \quad \int_C \frac{z+4}{z^2+2z+5}\,dz = 0 \text{ (By Cauchy Goursat Theorem)}$$

Ans.



Example 20. *Evaluate* $\int \frac{e^{-z}}{z+1}\,dz$ *where c is the circle* $|z| = \frac{1}{2}$.

Solution. The point $z = -1$ lies outside the circle $|z| = \frac{1}{2}$.

$\therefore$ The function $\frac{e^{-z}}{z+1}$ is analytic within and on C.

By Cauchy's Goursat theorem, we have $\oint_C \frac{e^{-z}}{z+1}\,dz = 0$. **Ans.**

Example 21. *Evaluate:* $\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\,dz$. *where C is the square with the vertices at* $1+i, 2+i, 2+2i, 1+2i$.

Solution. Here, $f(z) = \frac{2z^2+5}{(z+2)^3(z^2+4)}$

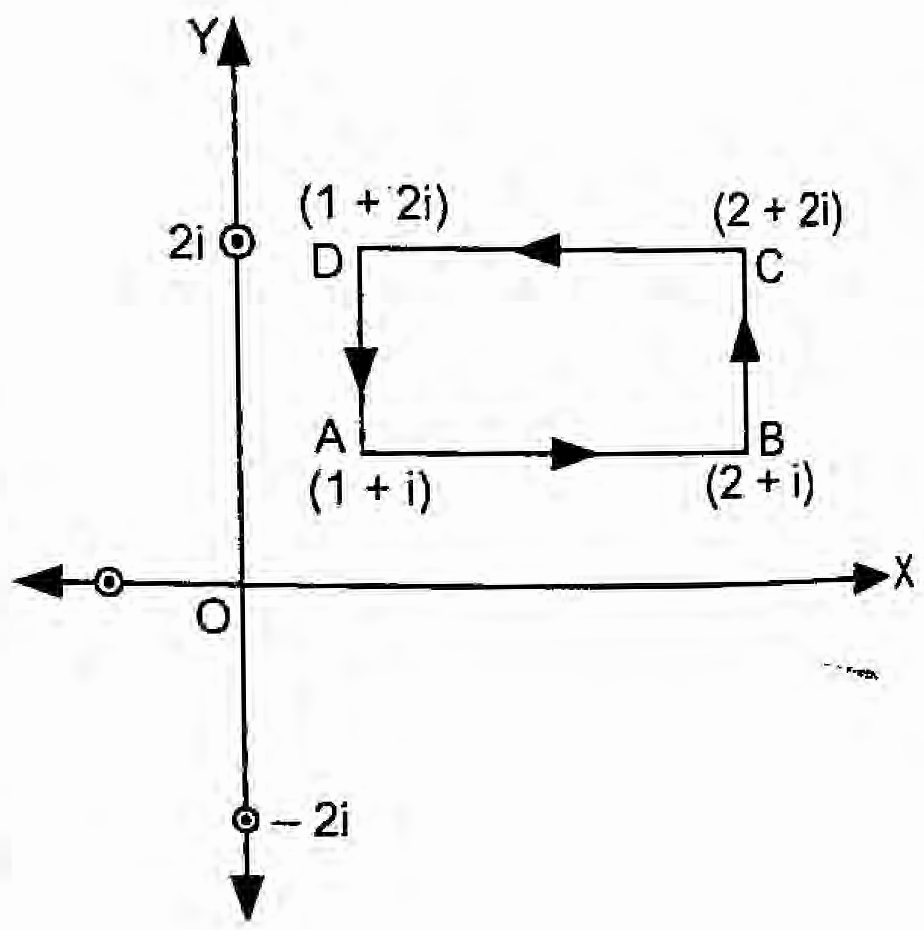
Poles are given by

$z = -2$ (pole of order 3)

$z = \pm 2i$ (simple poles).

Since the obtained poles do not lie inside the contour C with vertices at $1+i$, $2+i$, $2+2i$ and $1+2i$, hence by Cauchy Goursat theorem.

$$\oint_C \frac{2z^2+5}{(z+2)^3(z^2+4)}\,dz = 0 \quad \textbf{Ans.}$$

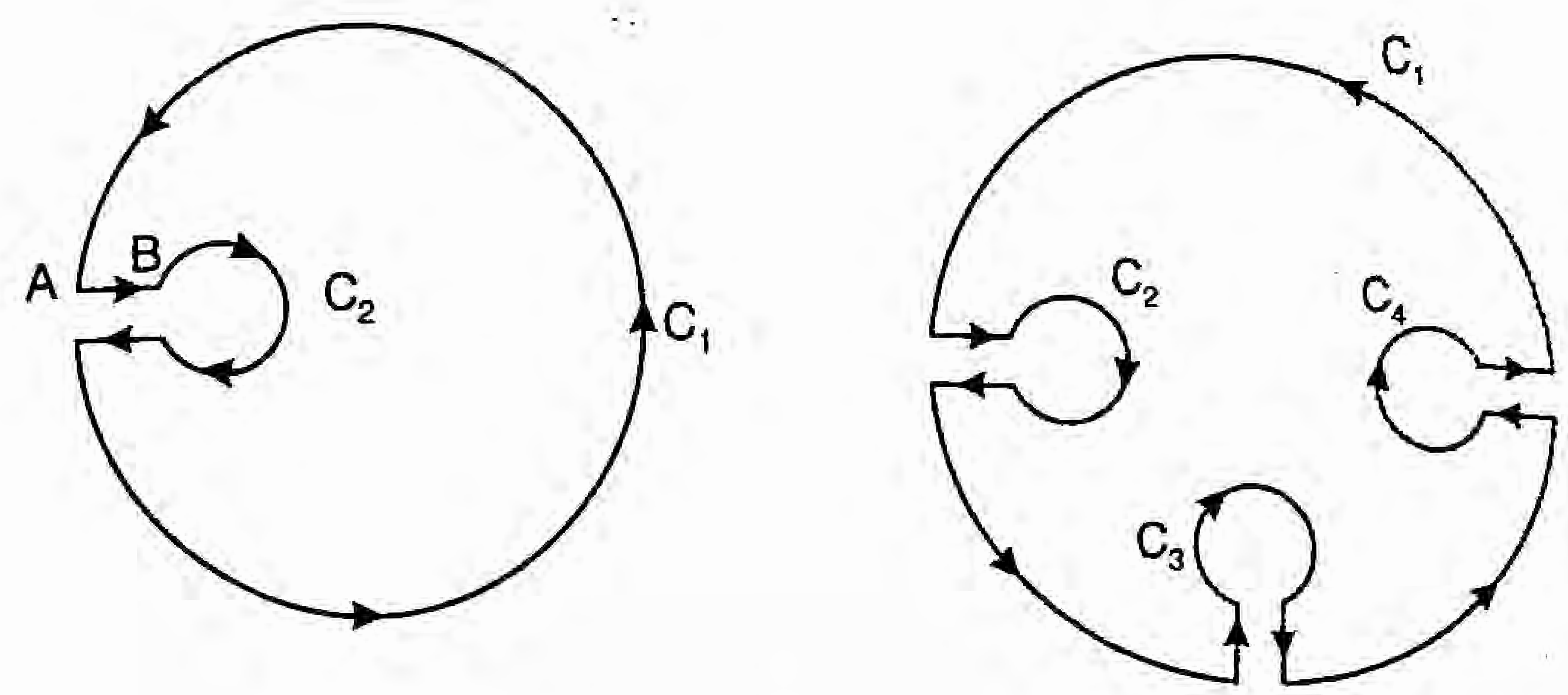


13.4 EXTENSION OF CAUCHY'S THEOREM TO MULTIPLE CONNECTED REGION

If $f(z)$ is analytic in the region R between two simple closed curves c_1 and c_2 then $\int_{c_1} f(z)dz = \int_{c2} f(z)dz$

Proof. $\int f(z)\,dz = 0$

where the path of integration is along AB, and curves C_2 in clockwise direction and along BA and along C_1 in anticlockwise direction.



$$\int_{AB} f(z)\,dz - \int_{c_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{c_1} f(z)dz = 0$$

$$\Rightarrow \quad -\int_{c_2} f(z)\,dz + \int_{c_1} f(z)\,dz = 0 \quad \text{as} \int_{AB} f(z)\,dz = -\int_{BA} f(z)\,dz$$

$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

Proved.

Corollary. $\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz + \int_{c_3} f(z)\,dz + \int_{c_4} f(z)\,dz$

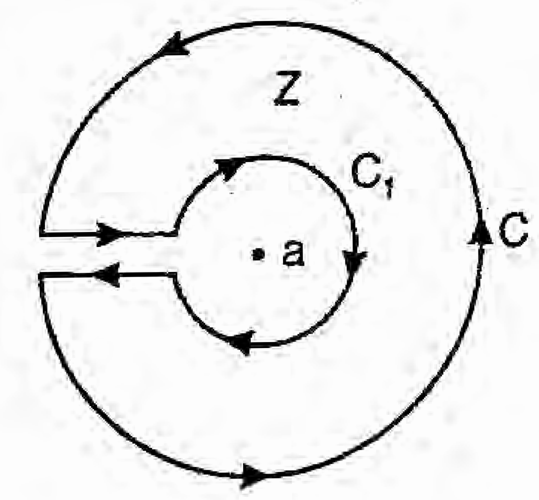
13.5 CAUCHY INTEGRAL FORMULA

(U.P., III Semester Dec. 2009)

If $f(z)$ is analytic within and on a closed curve C, and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$ *(R.G.P.V., Bhopal, III Semester, June 2008)*

Proof. Consider the function $\frac{f(z)}{z-a}$, which is analytic at all points within C, except $z = a$. With the point a as centre and radius r, draw a small circle C_1 lying entirely within C.



Now $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1; hence by Cauchy's Integral Theorem for multiple connected region, we have

$$\int_c \frac{f(z).dz}{z-a} = \int_{c_1} \frac{f(z)}{z-a}\,dz$$

$$= \int_{c_1} \frac{f(z) - f(a) + f(a)}{z - a} \, dz$$

$$= \int_{c_1} \frac{f(z) - f(a)}{z - a} dz + f(a) \int_{c_1} \frac{dz}{z - a} \quad \ldots (1)$$

For any point on C_1

Now, $\int_{c_1} \frac{f(z) - f(a)}{z - a} dz = \int_0^{2\pi} \frac{f(a + re^{i\theta}) - f(a)}{re^{i\theta}} ire^{i\theta} d\theta$ $[z - a = re^{i\theta}$ and $dz = ire^{i\theta} d\theta]$

$$= \int_0^{2\pi} [f(a + re^{i\theta}) - f(a)] \, id\theta = 0 \quad \text{(where } r \text{ tends to zero).}$$

$$\int_{c_1} \frac{dz}{z - a} = \int_0^{2\pi} \frac{ir e^{i\theta} d\theta}{r e^{i\theta}} = \int_0^{2\pi} id\theta = i[\theta]_0^{2\pi} = 2\pi i$$

Putting the values of the integrals in R.H.S. of (1), we have

$$\int_c \frac{f(z) \, dz}{z - a} = 0 + f(a) \, (2\pi i)$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z - a} dz$$

Proved.

13.6 CAUCHY INTEGRAL FORMULA FOR THE DERIVATIVE OF AN ANALYTIC FUNCTION

(R.G.P.V., Bhopal, III Semester, Dec. 2007)

If a function $f(z)$ is analytic in a region R, then its derivative at any point $z = a$ of R is also analytic in R, and is given by

$$f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - a)} dz$$

where c is any closed curve in R surrounding the point $z = a$.

Proof. We know Cauchy's Integral formula

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - a)} dz \quad \ldots (1)$$

Differentiating (1) w.r.t. 'a', we get

$$f'(a) = \frac{1}{2\pi i} \int_c f(z) \frac{\partial}{\partial a} \left(\frac{1}{z - a} \right) dz$$

$$\boxed{f'(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z - a)^2} dz}$$

Similarly,

$$\boxed{f''(a) = \frac{2!}{2\pi i} \int_c \frac{f(z) dz}{(z - a)^3}}$$

$$\boxed{f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z) dz}{(z - a)^{n+1}}}$$

13.7 MORERA THEOREM (Converse of Cauchy's Theorem)

If a function $f(z)$ is continuous in region D and if the integral $\int f(z) dz$, taken around any simple closed contour in D, is zero then $f(z)$ is an analytic function inside D.

Proof. $\int_{z_0}^{z} f(z) dz$ is independent of path from z_0 fixed point to a variable point z and hence must be function of z only. Thus $\int_{z_0}^{z} f(z) dz = \phi(z)$

$$\int (u + iv)(dx + idy) = U + iV \quad \text{and } f(z) = u + iv$$

$$\int_{(x_0, y_0)}^{(x, y)} (udx - vdy) = U \quad \text{and} \quad \int_{(x_0, y_0)}^{(x, y)} vdx + udy = V$$

Differentiating under the sign of integral, we get

$$\frac{\partial U}{\partial x} = u, \quad \frac{\partial V}{\partial x} = v, \quad \frac{\partial U}{\partial y} = -v, \quad \frac{\partial V}{\partial y} = u$$

$$\therefore \quad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} \quad \text{and} \quad \frac{\partial U}{\partial y} = \frac{-\partial V}{\partial x}$$

Thus, U and V satisfy $C - R$ equations.

$$\therefore \quad \phi(z) = U + iV \text{ is an analytic function.}$$

$$\phi'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = u + iv = f(z)$$

$f(z)$ is the derivative of an analytic function $\phi(z)$. **Proved.**

13.8 CAUCHY'S INEQUALITY

If $f(z)$ is analytic within a circle C *i.e.*, $|z - a| = R$ and if $|f(z)| \leq M$ on C, then

$$|f^n(a)| \leq \frac{Mn!}{R^n}$$

Proof. We know that $f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z) dz}{(z - a)^{n+1}} \leq \frac{n!}{|2\pi i|} \int_c \frac{|f(z)| \, |dz|}{|z - a|^{n+1}}$

$$\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} \int_0^{2\pi} R d\theta \quad [\text{since } z = R e^{i\theta}, |dz| = | iRe^{i\theta} d\theta| = Rd\theta]$$

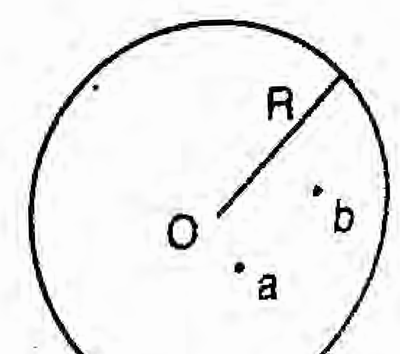
$$\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} 2\pi R$$

$$\leq \frac{Mn!}{R^n}$$

Proved.

Second Proof. Let a and b be any two points of z plane. Draw a large circle c_n with centre at the origin, of radius R, enclosing the points a and b. So that

$R > |a|$, and also $R > |b|$.



By Cauchy's integral formula

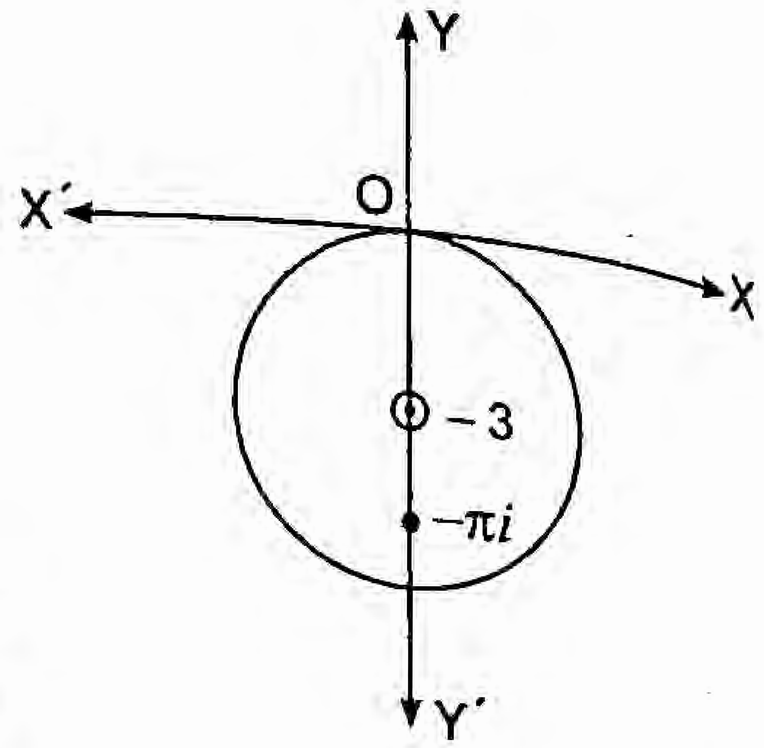
$$\int_c \frac{f(z) dz}{z - a} = 2\pi, \, if(a) \quad \text{and} \quad \int_c \frac{f(z)}{z - b} dz = 2\pi i$$

$$f(b) \, 2\pi i[f(a) - f(b)] = \int_c \frac{f(z) dz}{z - a} - \int_c \frac{f(z)}{z - b} dz$$

$$= \int_c \frac{a - b}{(z - a)(z - b)} f(z) \, dz$$

The given curve C is a circle with centre at $z = -3i$ i.e. at $(0, -3)$ and radius 1.

Clearly, only the pole $z = -\pi i$ lies inside the circle.



$$\therefore \quad \int_C \frac{dz}{z(z+\pi i)} = \int_C \frac{\frac{1}{z}}{z+\pi i}\, dz$$

$$= 2\pi i \left(\frac{1}{z}\right)_{z=-\pi i}$$

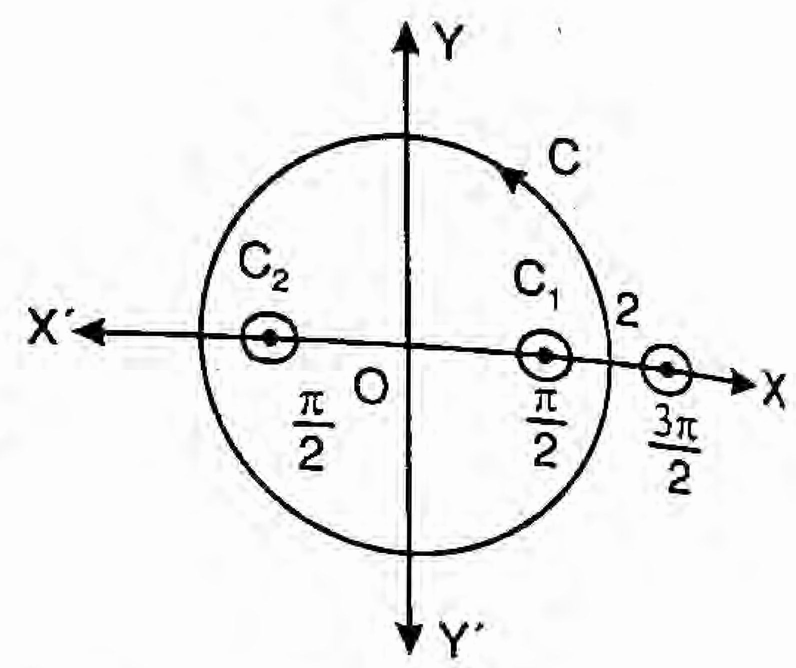
$$= \frac{2\pi i}{-\pi i} = -2$$

[By Cauchy's Integral formula]

Which is the required value of the given integral. **Ans.**

Example 29. *Evaluate the complex integral*

$$\int_c \tan z . dz \text{ where } c \text{ is } |z| = 2.$$



Solution. $\int_C \tan z . dz = \int_C \frac{\sin z}{\cos z} . dz$

$|z| = 2$, is a circle with centre at origin and radius = 2.

Poles are given by putting the denominator equal to zero.

$$\cos z = 0,\ z = -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

The integrand has two poles at $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ inside the given circle $|z| = 2$.

On applying Cauchy integral formula

$$\int_C \frac{\sin z}{\cos z} dz = \int_{C_1} \frac{\sin z}{\cos z} dz + \int_{C_2} \frac{\sin z}{\cos z} dz = 2\pi i\ [\sin z]_{z=\frac{\pi}{2}} + 2\pi i\ [\sin z]_{z=\frac{-\pi}{2}}$$

$$= 2\pi i(1) + 2\pi i(-1) = 0$$

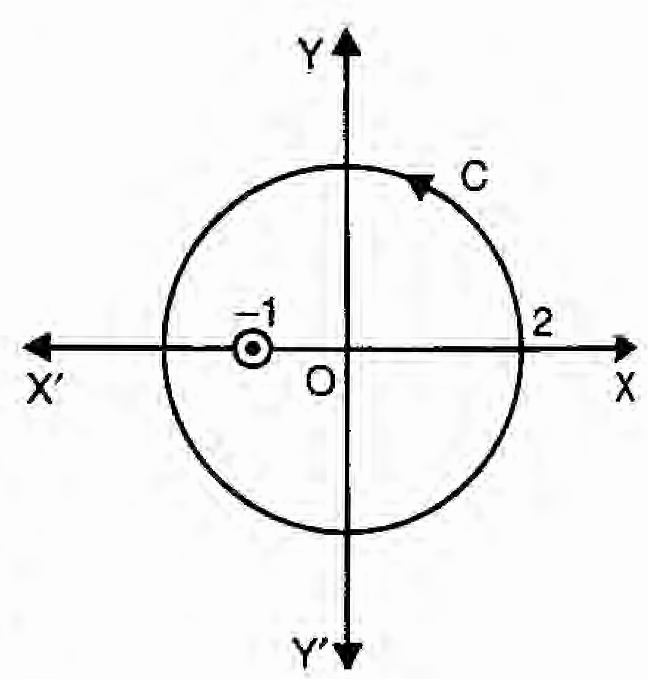
Which is the required value of the given integral. **Ans.**

Example 30. *Evaluate* $\oint_C \frac{e^{-z}}{z+1} dz$*, where C is the circle $|z| = 2$*

Solution. $f(z) = e^{-z}$ is an analytic function

The point $z = -1$ lies inside the circle $|z| = 2$.

$\therefore$ *By Cauchy's integral formula,*



$$\oint_C \frac{e^{-z}}{z+1} dz = 2\pi i (e^{-z})_{z=-1} = 2\pi i e. \quad \textbf{Ans.}$$

Example 31. *Evaluate:* $\int_C \frac{e^z}{(z-1)(z-4)} dz$ *where C is the circle $|z| = 2$ by using Cachy's Integral Formula.*

(R.G.P.V., Bhopal, III Semester, June 2006)

Solution. We have,

$\int_C \frac{e^z}{(z-1)(z-4)} dz$ where C is the circle with centre at origin and radius 2.

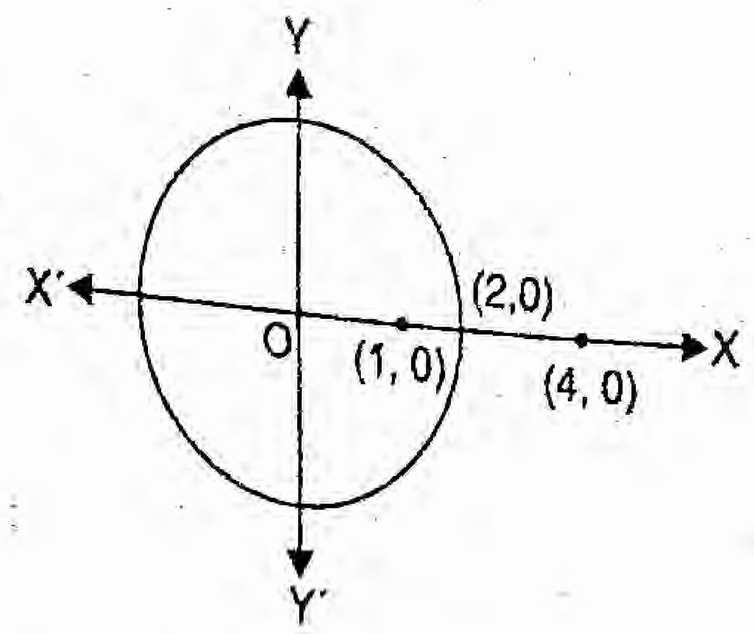
Poles are given by putting the denominator equal to zero.

$$(z-1)(z-4) = 0$$

$$\Rightarrow \quad z = 1, 4$$

Here there are two simple poles at $z = 1$ and $z = 4$.

There is only one pole at $z = 1$ inside the contour.



Therefore

$$\int_C \frac{e^z}{(z-1)(z-4)} dz = \int \frac{\frac{e^z}{(z-4)}}{(z-1)} dz$$

$$= 2\pi i \left[\frac{e^z}{z-4}\right]_{z=1} \quad \text{(By Cauchy Integral Theorem)}$$

$$= 2\pi i \left[\frac{e}{1-4}\right]$$

$$= -\frac{2\pi i e}{3}$$

Which is the required value of the given integral. **Ans.**

Example 32. *If $f(z_1) = \int_C \frac{3z^2+7z+1}{z-z_1} dz$, where C is the circle $x^2 + y^2 = 4$, find the values of*

(i) $f(3)$, *(ii) $f'(1-i)$,* *(iii) $f''(1-i)$.*

Solution. The given circle C is $x^2 + y^2 = 4$ or $|z| = 2$.

The point $z = 3$ lies outside the circle $|z| = 2$.

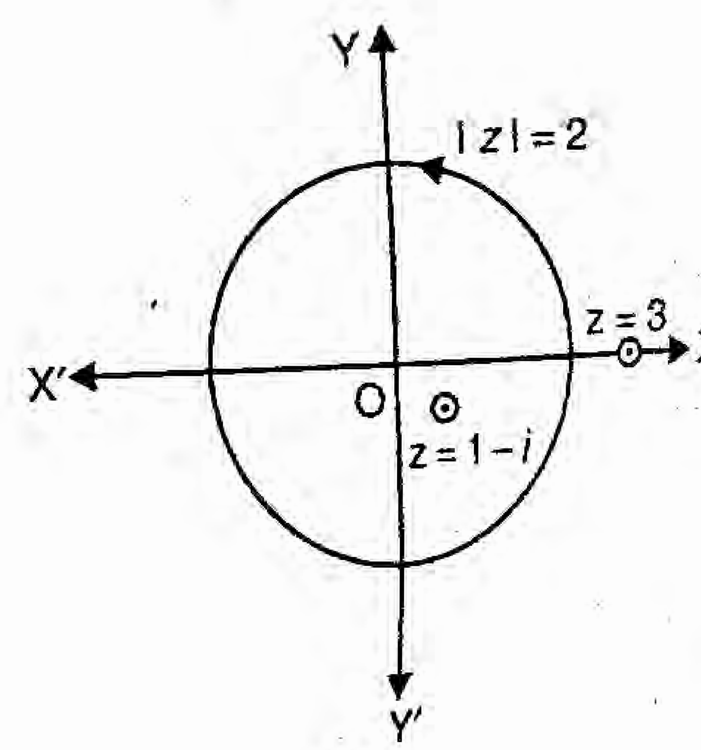
(*i*) $f(3) = \oint_C \frac{3z^2+7z+1}{z-3} dz$ and $\frac{3z^2+7z+1}{z-3}$ is analytic within and on C.

$\therefore$ By Cauchy's integral theorem, we have

$$\oint_C \frac{3z^2+7z+1}{z-3} dz = 0 \Rightarrow f(3) = 0. \quad \textbf{Ans.}$$

(*ii*) $z_1 = 1 - i$ lies inside the circle C.

By Cauchy's Integral formula, we have



$$\int_C \frac{3z^2+7z+1}{z-z_1} dz = 2\pi i\ (3z^2+7z+1)_{z=z_1}$$

$$f(z) = 2\pi i\ (3z^2 + 7z + 1)$$

$$f'(z) = 2\pi i\ (6z + 7)$$

$$f'(1-i) = 2\pi i\ [6(1-i) + 7]$$

$$\Rightarrow \quad f'(1-i) = 2\pi i\ [13 - 6i]$$

$$\Rightarrow \quad f'(1-i) = 2\pi\ [6 + 13i] \quad \textbf{Ans.}$$

(*iii*) $f''(z) = 2\pi i\ 6$

$$f''(1-i) = 12\pi i \quad \textbf{Ans.}$$

Example 33. *Evaluate*

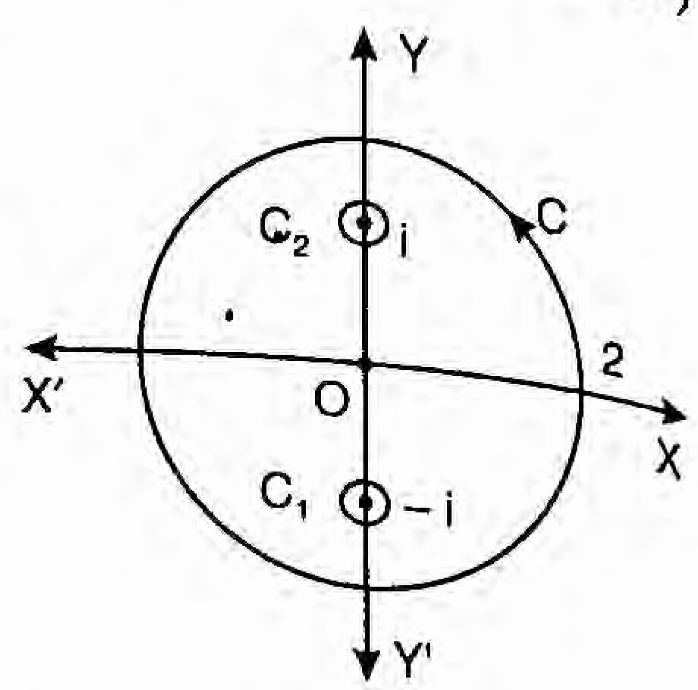
$$\int_c \frac{e^z}{z^2+1}\,dz \text{ over the circular path } |z| = 2. \text{ (U.P., III Semester, Dec. 2009)}$$

(AKTU, 2016-2017)

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$$z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$$

The integrand has two simple poles at $z = i$ and $z = -i$. Both poles are inside the given circle with centre at origin and radius 2.



$$\int_c \frac{1}{2i}\left(\frac{e^z}{z-i} - \frac{e^z}{z+i}\right) dz = \int_c \frac{1}{2i}\frac{e^z}{z-i}\,dz - \frac{1}{2i}\int_c \frac{e^z}{z+i}\,dz$$

$$= \frac{1}{2i}\left[2\pi i (e^z)_{z=i} - 2\pi i (e^z)_{z=-i}\right]$$

$$= \frac{2\pi i}{2i}[e^i - e^{-i}] = \pi\,(2\; i \sin 1) = 2\pi i \sin 1$$

Which is the required value of the given integral. **Ans.**

Second Method.
$$\int_C \frac{e^z}{z^2+1}\,dz = \int_C \frac{e^z\,dz}{(z+i)(z-i)} = \int_{C_1} \frac{\frac{e^z}{z-i}}{z+i}\,dz + \int_{C_2} \frac{\frac{e^z}{z+i}}{z-i}\,dz$$

$$= 2\pi i \left(\frac{e^z}{z-i}\right)_{z=-i} + 2\pi i \left(\frac{e^z}{z+i}\right)_{z=i}$$

$$= \left[2\pi i \frac{e^{-i}}{-i-i} + 2\pi i \frac{e^i}{i+i}\right] = \pi[-e^{-i} + e^i]$$

$$= \pi(2i \sin 1) = 2\pi i \sin 1$$

Which is the required value of the given integral. **Ans.**

Example 34. *Evaluate* $\int_c \frac{dz}{z^2-1}$, *where C is the circle* $x^2 + y^2 = 4$.

Solution. Poles are given by putting the denominator equal to zero.

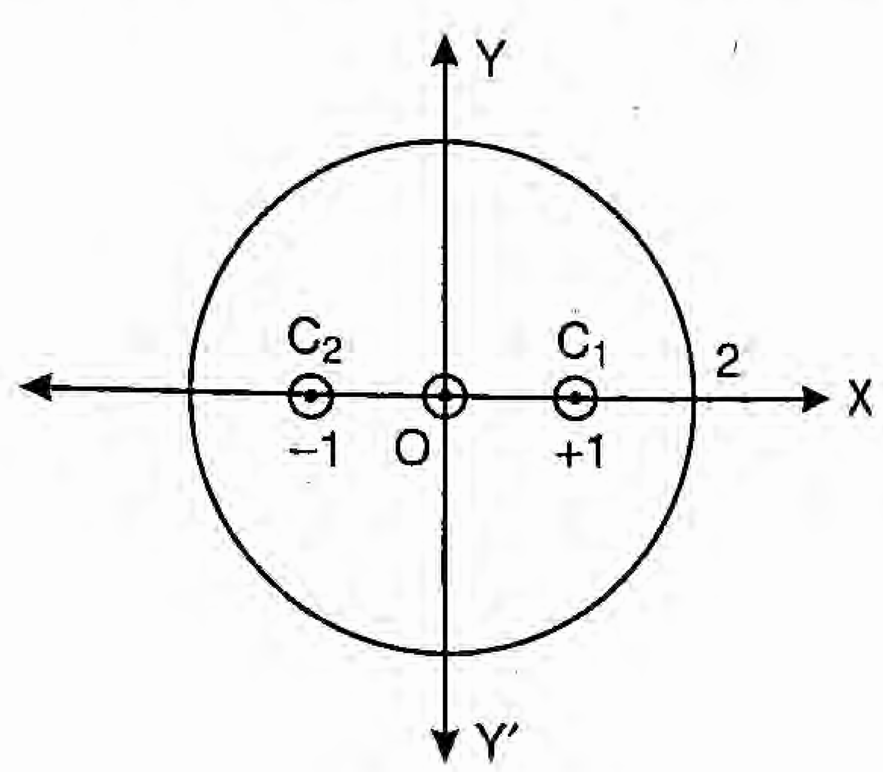
$$z^2 - 1 = 0,\; z^2 = 1,\; z = \pm 1$$

The given circle $x^2 + y^2 = 4$ with centre at $z = 0$ and radius 2 encloses two simple poles at $z = 1$ and $z = -1$

$$\therefore \int_C \frac{dz}{z^2-1} = \int_{C_1} \frac{dz}{z^2-1} + \int_{C_2} \frac{dz}{z^2-1}$$

$$= \int_{C_1} \frac{\frac{1}{z+1}}{z-1}\,dz + \int_{C_2} \frac{\frac{1}{z-1}}{z+1}\,dz$$

$$= 2\pi i \left(\frac{1}{z+1}\right)_{z=1} + 2\pi i \left(\frac{1}{z-1}\right)_{z=-1}$$



$$= 2\pi i \left[\frac{1}{1+1}\right] + 2\pi i \left[\frac{1}{-1-1}\right]$$

$$= \pi i - \pi i = 0$$

Which is the required value of the given integral. **Ans.**

Example 35. *Evaluate* $\int_C \frac{z}{z^2+1}\,dz$ *where*

(i) *C is* $|z + 1/z| = 2$

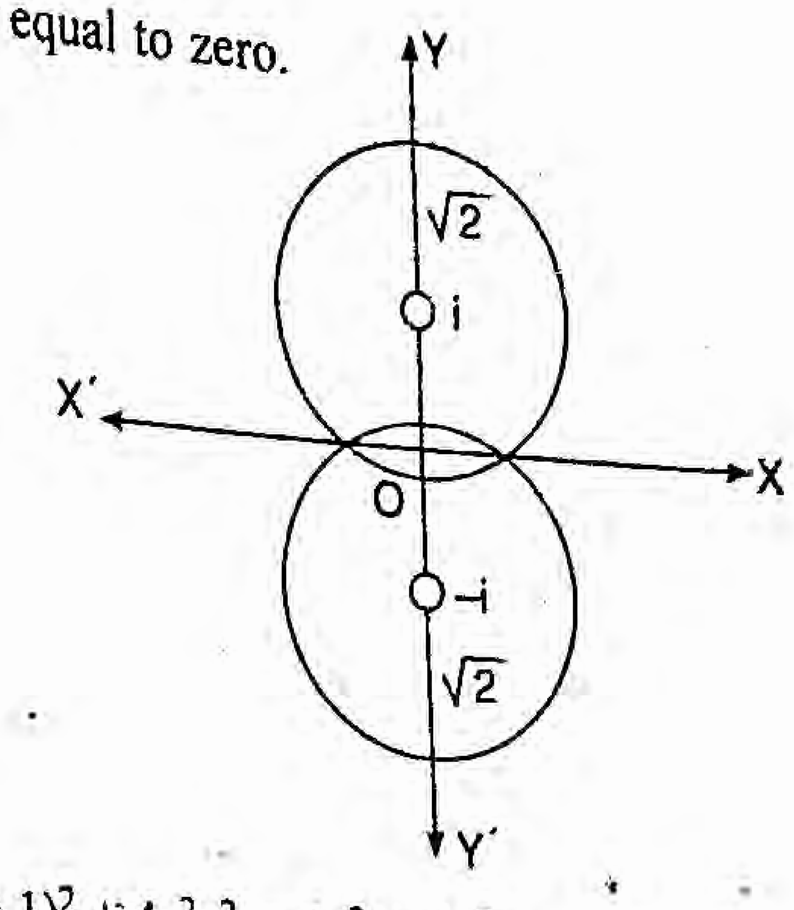
(ii) *C is* $|z + i| = 1$.

Solution. Poles are found by putting the denominator equal to zero.

$z^2 + 1 = 0$ or $z^2 = -1$ or $z = \pm i$

The integrand has two *poles at* $z = +i$, $z = -i$

(i) $\left|z + \frac{1}{z}\right| = 2$ is the given curve



$$\Rightarrow \left|x + iy + \frac{1}{x+iy}\right| = 2$$

$$\Rightarrow \left|\frac{x^2 - y^2 + 2ixy + 1}{x+iy}\right| = 2$$

$$\Rightarrow \frac{(x^2 - y^2 + 1)^2 + 4x^2y^2}{x^2 + y^2} = 4 \text{ or } (x^2 - y^2 + 1)^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow x^4 + y^4 - 2x^2y^2 + 1 + 2x^2 - 2y^2 + 4x^2y^2 = 4x^2 + 4y^2$$

$$\Rightarrow x^4 + y^4 + 2x^2y^2 - 2(x^2 + y^2) - 4y^2 + 1 = 0$$

$$\Rightarrow (x^2 + y^2)^2 - 2(x^2 + y^2) + 1 = 4y^2 \Rightarrow (x^2 + y^2 - 1)^2 = (2y)^2$$

$$\Rightarrow x^2 + y^2 - 1 = \pm 2y \Rightarrow x^2 + y^2 \pm 2y - 1 = 0 \Rightarrow x^2 + (y \pm 1)^2 = 2$$

This equation represents two circles with centres (0, 1), (0, –1) and radius $\sqrt{2}$.

$$\int_C \frac{z}{z^2+1}\,dz = \int_{C_1} \frac{z}{z^2+1}\,dz + \int_{C_2} \frac{z}{z^2+1}\,dz$$

$$= \int_{C_1} \frac{\frac{z}{z+i}}{z-i}\,dz + \int_{C_2} \frac{\frac{z}{z-i}}{z+i}\,dz$$

$$= 2\pi i \left(\frac{z}{z+i}\right)_{z=i} + 2\pi i \left(\frac{z}{z-i}\right)_{z=-i}$$

$$= 2\pi i \left(\frac{i}{i+i} + \frac{-i}{-i-i}\right) = 2\pi i \left[\frac{1}{2} + \frac{1}{2}\right]$$

$$= 2\pi i$$

Which is the required value of the given integral. **Ans.**

(ii) $|z + i| = 1$ is a circle with centre at $z = -i$ and its radius is 1.

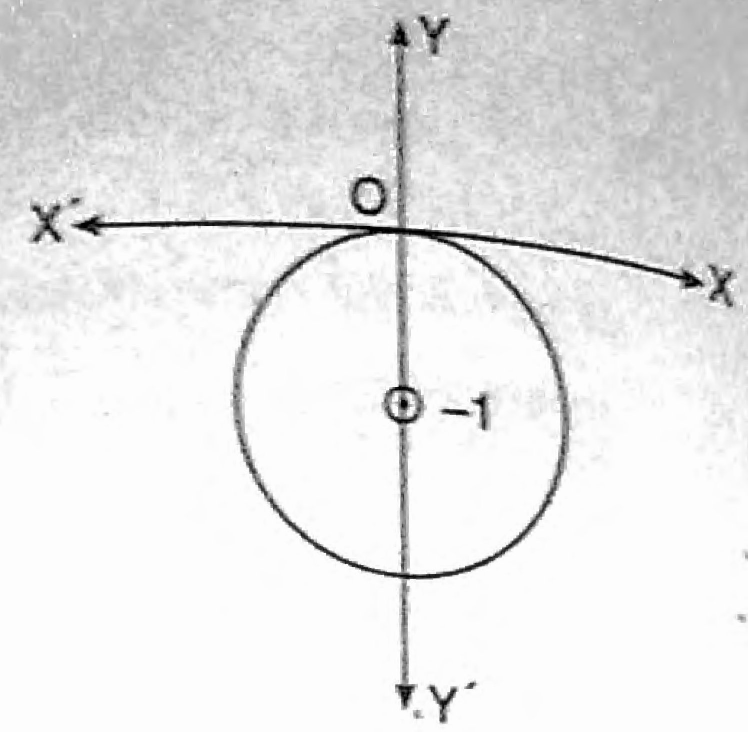
The integrand has a simple pole at $z = -i$

$$\int_C \frac{z}{z^2+1}\,dz = \int_C \frac{\frac{z}{z-i}}{z+i}\,dz$$

$$= 2\pi i \left(\frac{z}{z-i}\right)_{z=-i}$$

$$= 2\pi i \left(\frac{-i}{-i-i}\right)$$

$$= 2\pi i \left(\frac{1}{2}\right) = \pi i$$



Which is the required value of the given integral. **Ans.**

Example 36. *Evaluate the following integral using Cauchy integral formula*

$$\int_c \frac{4-3z}{z(z-1)(z-2)}\,dz \text{ where } c \text{ is the circle } |z| = \frac{3}{2}.$$

(R.G.P.V., Bhopal, III Semester, June 2008)

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$z(z-1)(z-2) = 0$ or $z = 0, 1, 2$

The integrand has three simple poles at $z = 0, 1, 2$.

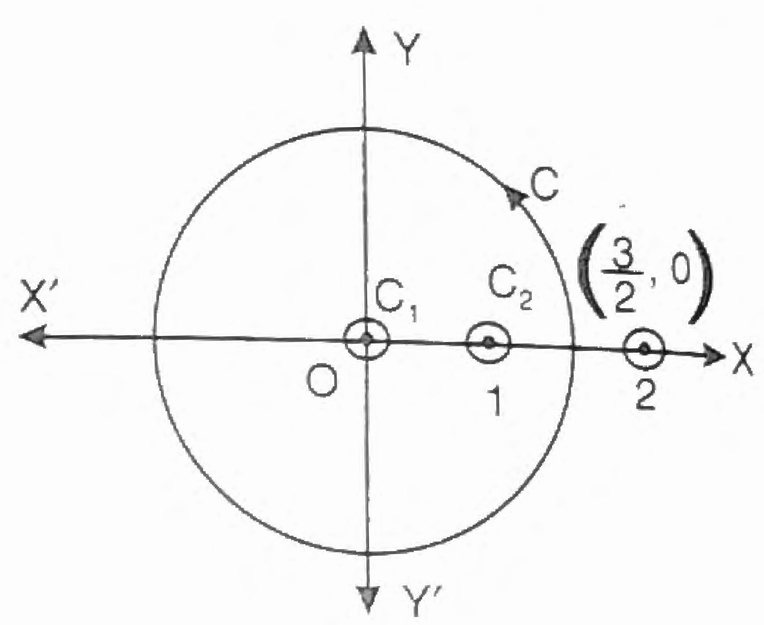
The given circle $|z| = \frac{3}{2}$ with centre at $z = 0$ and radius $= \frac{3}{2}$ encloses two poles $z = 0$, and $z = 1$.

$$\int_C \frac{4-3z}{z(z-1)(z-2)}\,dz$$

$$= \int_{c_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{c_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

$$= 2\pi i \left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0} + 2\pi i \left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \cdot \frac{4}{(-1)(-2)} + 2\pi i \frac{4-3}{1(1-2)} = 2\pi i(2-1) = 2\pi i$$



Which is the required value of the given integral. **Ans.**

Example 37. *Evaluate* $\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$ *where c is the circle* $|z| = 10$.

(U.P. III Semester, June 2009)

Solution. Here, we have $\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz$

The poles are determined by putting the denominator equal to zero.

i.e.; $(z+1)^2(z^2+4) = 0 \Rightarrow z = -1, -1$ and $z = \pm 2i$

The circle $|z| = 10$ with centre at origin and radius $= 10$.

encloses a pole at $z = -1$ of second order and simple poles $z = \pm 2$ i

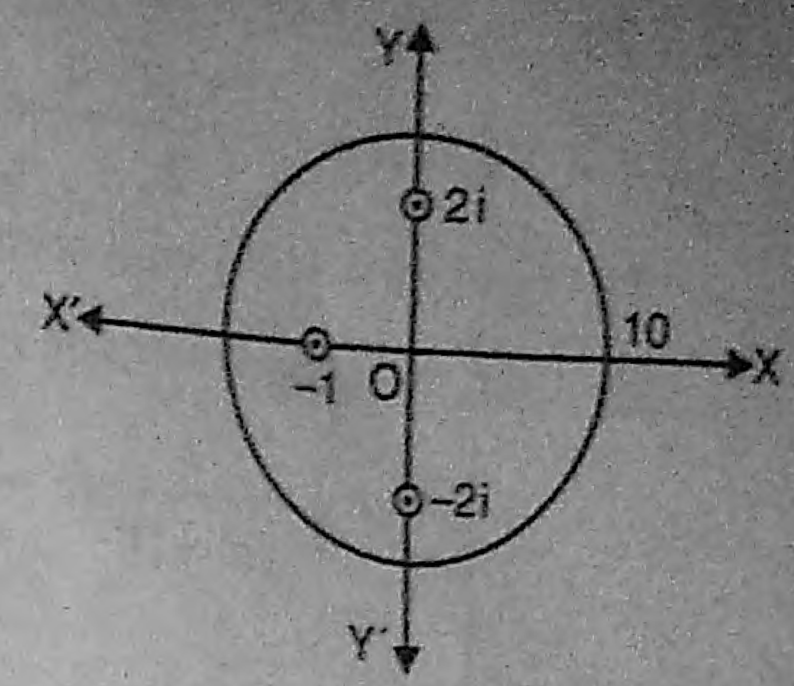
The given integral $= I_1 + I_2 + I_3$

$$I_1 = \int_{c_1} \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = \int_{c_1} \frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2}\,dz$$

$$= 2\pi i \left[\frac{d}{dz}\frac{z^2-2z}{z^2+4}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(z^4+4)(2z-2)-(z^2-2z)\,2z}{(z^2+4)^2}\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(1+4)(-2-2)-(1+2)\,2(-1)}{(1+4)^2}\right] = 2\pi i\left(-\frac{14}{25}\right) = \frac{-28\pi i}{25}$$



$$I_2 = \int_{c_2} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}\,dz = 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i} = 2\pi i \left[\frac{-4-4i}{(2i+1)^2(2i+2i)}\right]$$

$$= 2\pi i \frac{(1+i)}{4+3i}$$

$$I_3 = \int_{c_3} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)}\,dz = 2\pi i \left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i \left[\frac{-4+4i}{(-2i+1)^2(-2i-2i)}\right] = 2\pi i \frac{(i-1)}{(3i-4)}$$

$$\int_c \frac{z^2-2z}{(z+1)^2(z^2+4)}\,dz = I_1 + I_2 + I_3$$

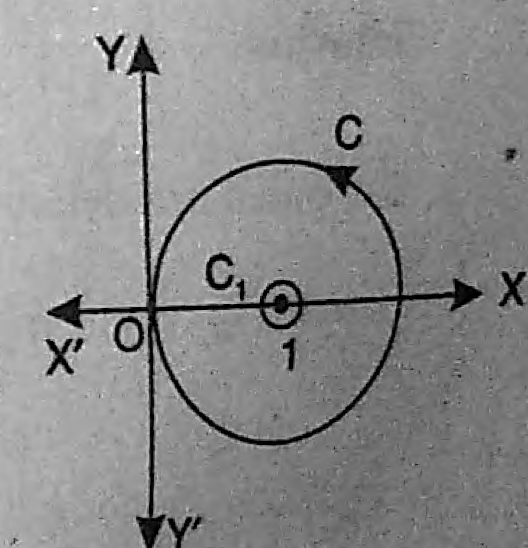
$$= \frac{-28\pi i}{25} + 2\pi i\left(\frac{1+i}{4+3i}\right) + 2\pi i\left(\frac{i-1}{3i-4}\right)$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{1+i}{(4+3i)} + \frac{(i-1)}{(3i-4)}\right]$$

$$= 2\pi i \left[\frac{-14}{25} + \frac{(1+i)(3i-4)+(i-1)(4+3i)}{(-9-16)}\right]$$

$$= \frac{2\pi i}{-25}\,[14 + (3i-4-3-4i) + (4i-3-4-3i)] = 0$$ **Ans.**

Example 38. *Integrate* $\frac{1}{(z^3-1)^2}$ *the counter clock-wise sense around the circle* $|z-1| = 1$.



Solution. Poles of the given function are found by putting denominator equal to zero.

$(z^3-1)^2 = 0,$

$$(z-1)^2(z^2+z+1)^2=0$$

$$z=1,1,\ z=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{-3}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$$

The circle $|z-1|=1$ with centre at $z=1$ and unit radius encloses a pole of order two at $z=1$.

By Cauchy Integral formula

$$\int_C \frac{1}{(z^3-1)^2}dz=\int_{C_1}\frac{1}{(z-1)^2(z^2+z+1)^2}dz$$

$$=\int_{C_1}\frac{\frac{1}{(z^2+z+1)^2}}{(z-1)^2}dz$$

$$=2\pi i\left[\frac{d}{dz}\frac{1}{(z^2+z+1)^2}\right]_{z=1}=2\pi i\left[\frac{-2(2z+1)}{(z^2+z+1)^3}\right]_{z=1}$$

$$=2\pi i\left[\frac{-2(2+1)}{(1+1+1)^3}\right]=-\frac{4\pi i}{9}$$ **Ans.**

Example 39. *Find the value of* $\int_C \frac{3z^2+z}{z^2-1}dz$.

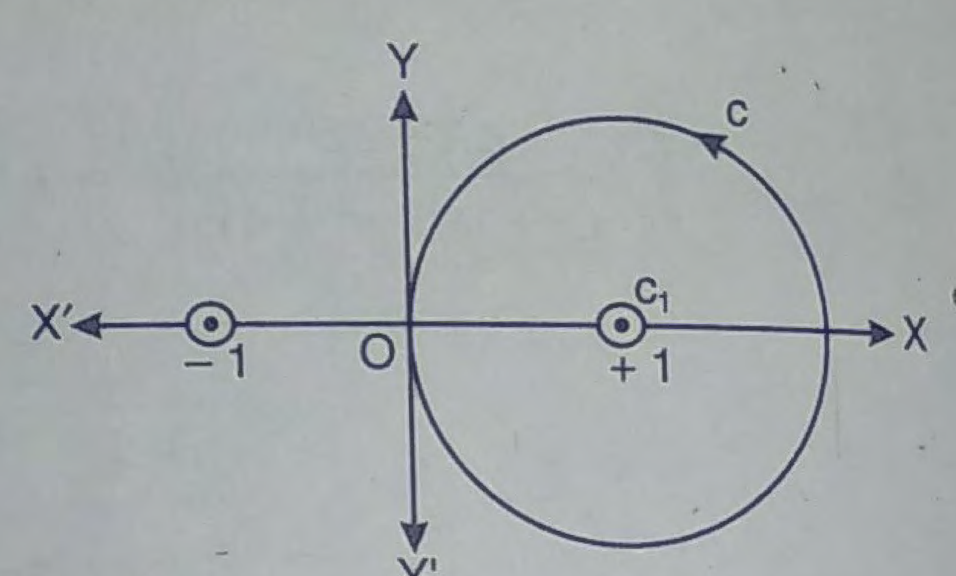
If c is circle $|z-1|=1$ *(R.G.P.V., Bhopal, III Semester, June 2007)*

Solution. Poles of the integrand are given by putting the denominator equal to zero.

$$z^2-1=0,\ z^2=1,\ z=\pm 1$$

The circle with centre $z=1$ and radius unity encloses a simple pole at $z=1$.

By Cauchy Integral formula



$$\int_C \frac{3z^2+z}{z^2-1}dz=\int_C\frac{\frac{3z^2+z}{z+1}}{z-1}dz$$

$$=2\pi i\left[\frac{3z^2+z}{z+1}\right]_{z=1}=2\pi i\left(\frac{3+1}{1+1}\right)=4\pi i$$

Which is the required value of the given integral. **Ans.**

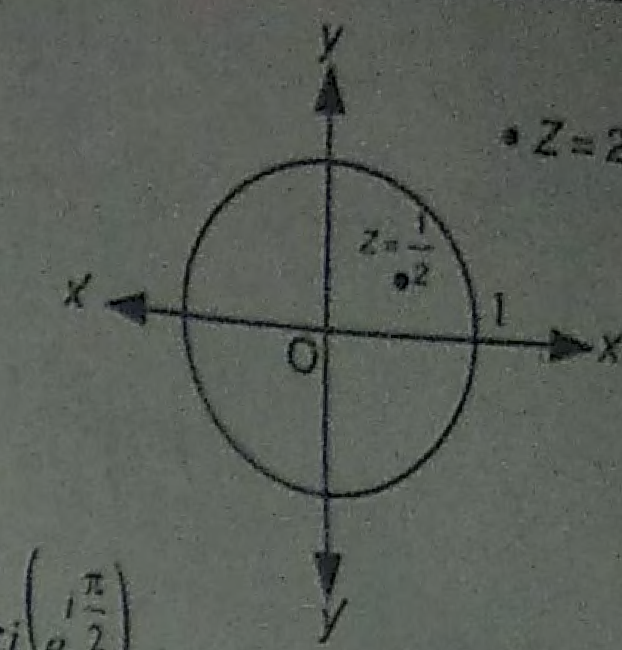
Example 40. *Find the value of the integral* $\oint_C \frac{\exp(i\pi z)}{2z^2-5z+2}dz$, *where C is the unit circle with centre at the origin.* *(G.B.T.U., III Sem., April 2012)*

Solution. $\oint_C \frac{\exp(i\pi z)}{2z^2-5z+2}dz$

The poles are determined by putting the denominator equal to zero.

$$2z^2-5z+2=0\Rightarrow(2z-1)(z-2)=0\Rightarrow z=\frac{1}{2}\text{ and }z=2$$

There is only one pole at $z=\frac{1}{2}$ inside the unit circle.



$$\oint_C \frac{e^{i\pi z}}{(2z-1)(z-2)}dz=\oint_C\frac{\frac{e^{i\pi z}}{(z-2)}}{(2z-1)}dz$$

$$=\frac{1}{2}\oint\frac{\frac{e^{i\pi z}}{(z-2)}}{\left(z-\frac{1}{2}\right)}dz=(2\pi i)\frac{1}{2}\left[\frac{e^{i\pi z}}{z-2}\right]_{z=\frac{1}{2}}=\pi i\frac{e^{\frac{i\pi}{2}}}{\frac{1}{2}-2}=-\frac{2}{3}\pi i\left(e^{i\frac{\pi}{2}}\right)$$

$$=-\frac{2\pi i}{3}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=-\frac{2\pi i}{3}(i)=\frac{2\pi}{3}$$ **Ans.**

Example 41. *Evaluate* $\oint_C \frac{z^2+1}{z^2-1}dz$ *where c is circle,*

(*i*) $|z|=\frac{3}{2}$

(*ii*) $|z^2-1|=1$,

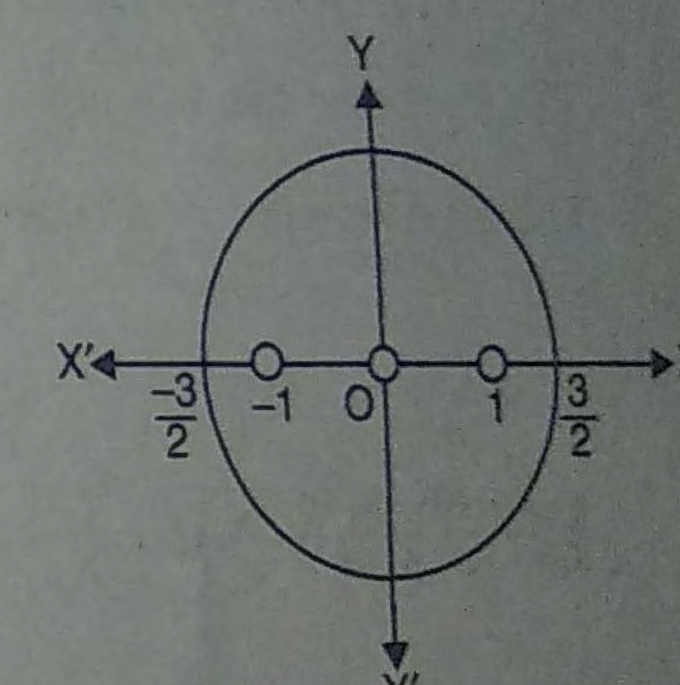
(*iii*) $|z|=\frac{1}{2}$.

Solution. Poles of the integrand are given by putting the denominator equal to zero.

i.e.; $z^2-1=0\ \Rightarrow\ z=1,-1$

(*i*) $|z|=\frac{3}{2}$ is equation of circle C with centre O and radius $\frac{3}{2}$.

Both poles $z=1,-1$ lie inside C.



$$\oint_C\frac{z^2+1}{z^2-1}dz=\oint_{C_1}\frac{\left(\frac{z^2+1}{z-1}\right)}{z+1}dz+\oint_{C_2}\frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}dz$$

$$=2\pi i\left[\frac{z^2+1}{z-1}\right]_{z=-1}+2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1}$$

$$=2\pi i\left[\frac{1+1}{-1-1}\right]+2\pi i\left[\frac{1+1}{1+1}\right]$$

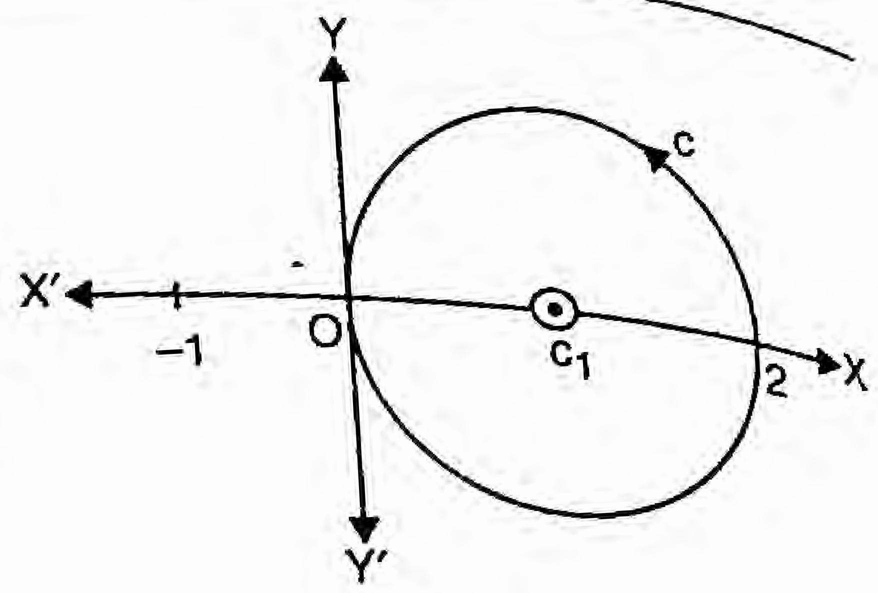
$$=-2\pi i+2\pi i=0$$

Which is the required value of the given integral. **Ans.**

(*ii*) $|z-1|=1$ is equation of circle C with centre 1 and radius 1 encloses only pole $z=1$.

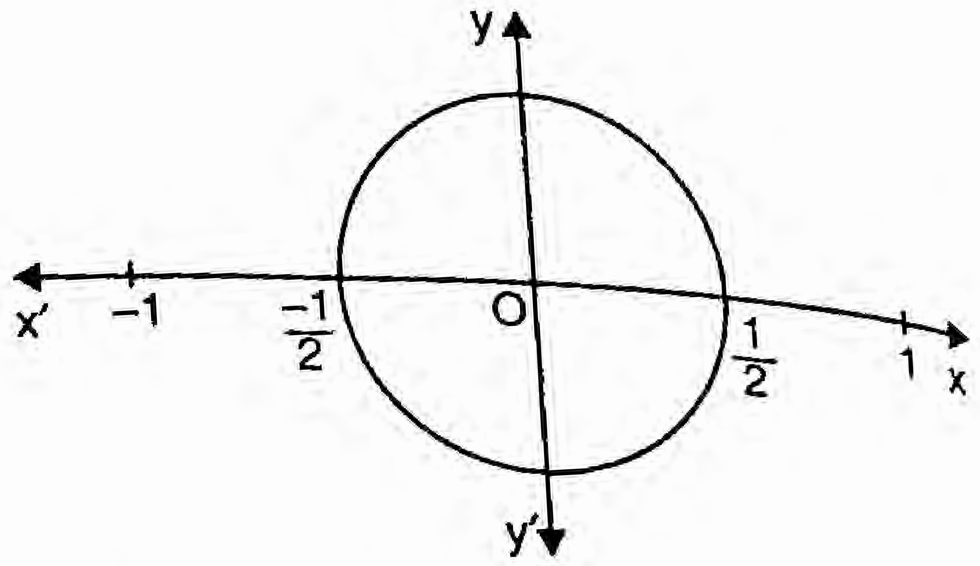
$$\oint_C \frac{z^2+1}{z^2-1}\,dz = \oint_{C_1} \frac{\left(\frac{z^2+1}{z+1}\right)}{z-1}\,dz.$$

$$= 2\pi i\left[\frac{z^2+1}{z+1}\right]_{z=1} \quad z= 2\pi i\left[\frac{1+1}{1+1}\right] = 2\pi i$$



(*iii*) $|z| = \frac{1}{2}$ is equation of circle C with centre O

and radius $\frac{1}{2}$ There is no pole inside C.

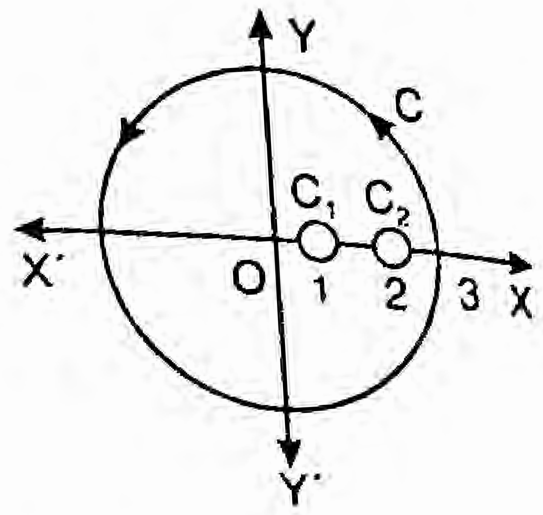


Hence, $\oint_C \frac{z^2+1}{z^2-1}\,dz = 0$. **Ans.**

Example 42. *Use Cauchy integral formula to evaluate.*

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$$

where c is the circle $|z| = 3$.



Solution. $\oint \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$

Poles of the integrand are given by putting the denominator equal to zero.

$(z-1)(z-2) = 0 \Rightarrow z = 1, 2$

The integrand has two poles at $z = 1, 2$.

The given circle $|z| = 3$ with centre at $z = 0$ and radius 3 encloses both the poles $z = 1$, and $z = 2$.

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz = \int_{C_1} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)}}{(z-1)}\,dz + \int_{C_2} \frac{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)}}{(z-2)}\,dz$$

$$= 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\right]_{z=1} + 2\pi i\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]_{z=2}$$

$$= 2\pi i\left(\frac{\sin \pi + \cos \pi}{1-2}\right) + 2\pi i\left(\frac{\sin 4\pi + \cos 4\pi}{2-1}\right)$$

$$= 2\pi i\left(\frac{-1}{-1}\right) + 2\pi i\left(\frac{1}{1}\right) = 4\pi i$$

Which is the required value of the given integral. **Ans.**

Example 43. *Let* $P(z) = a + bz + cz^2$ *and*

$$\oint_C \frac{P(z)}{z}\,dz = \oint_C \frac{P(z)}{z^2}\,dz = \oint_C \frac{P(z)}{z^3}\,dz = 2\pi i$$

where C is the circle $|z| = 1$. *Evaluate* $P(z)$.

Solution. (*i*) $\oint_C \frac{P(z)}{z}\,dz = 2\pi i$

Here, $z = 0$ is a simple pole which lies inside the circle $|z| = 1$

$$\oint_C \frac{P(z)}{z}\,dz = 2\pi i\,[P(z)]_{z=0} \qquad \text{(By Cauchy's Integral formula)}$$

From (1) and (2), we have

$\Rightarrow \quad 2\pi i\,(a + bz + cz^2)_{z=0} = 2\pi i$

$\Rightarrow \quad 2\pi i(a) = 2\pi i$

$\Rightarrow \quad a = 1$

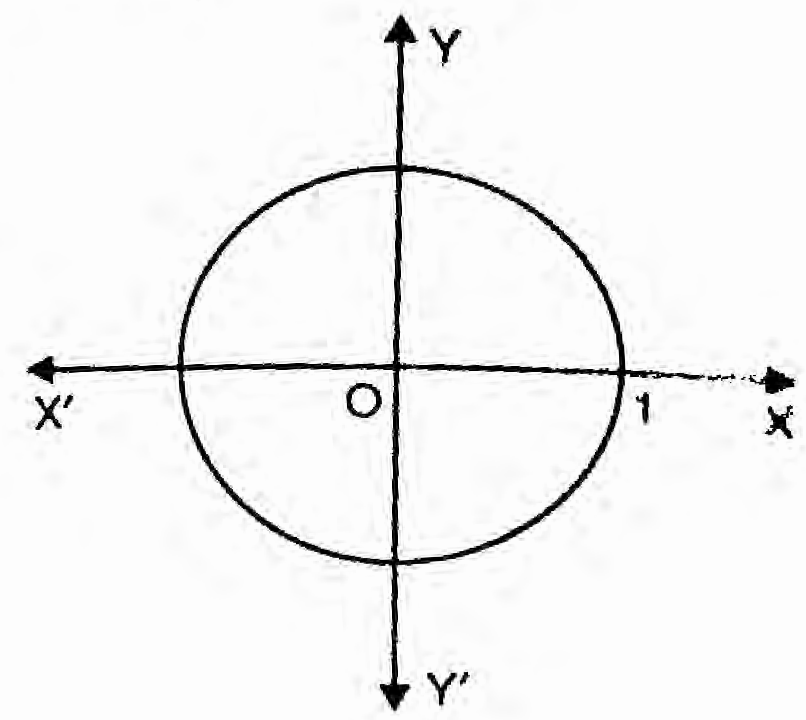
(*ii*) $\oint_C \frac{P(z)}{z^2}\,dz = 2\pi i$ ...(3)

Here, $z = 0$ is a double pole which lies inside the circle $|z| = 1$

$$\oint_C \frac{P(z)}{z^2}\,dz = \frac{2\pi i}{1!}\left\{\frac{d}{dz}P(z)\right\}_{z=0}$$

$$= 2\pi i\left\{\frac{d}{dz}(a + bz + cz^2)\right\}_{z=0}$$

$$= 2\pi i\,(b + 2cz)_{z=0}$$



$\Rightarrow \quad \oint_C \frac{P(z)}{z^2}\,dz = 2\pi i b$ ...(4)

From (3) and (4), we get

$2\pi i(b) = 2\pi i$

$\Rightarrow \quad b = 1$

(*iii*) $\oint_C \frac{P(z)}{z^3}\,dz = 2\pi i$ ...(5)

Here, $z = 0$ is a pole of order three which lies inside the circle $|z| = 1$.

$$\oint_C \frac{P(z)}{z^3}\,dz = \frac{2\pi i}{2!}\left\{\frac{d^2}{dz^2}P(z)\right\}_{z=0}$$

$$= \frac{2\pi i}{2}\left\{\frac{d^2}{dz^2}(a + bz + cz^2)\right\}_{z=0} = \pi i\,(2c)$$

$\Rightarrow \quad \oint_C \frac{P(z)}{z^3}\,dz = 2\pi i(c)$ ...(6)

From (5) and (6), we get

$2\pi i(c) = 2\pi i$

$\Rightarrow \quad c = 1$

Putting the values of a, b and c in $P(z)$, we get

$P(z) = 1 + z + z^2$.

Which is the required value of the given integral. **Ans.**

Example 44. *Evaluate the following complex integration using Cauchy's integral formula*

$$\int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$$

where C is the circle $|z| = 2$.

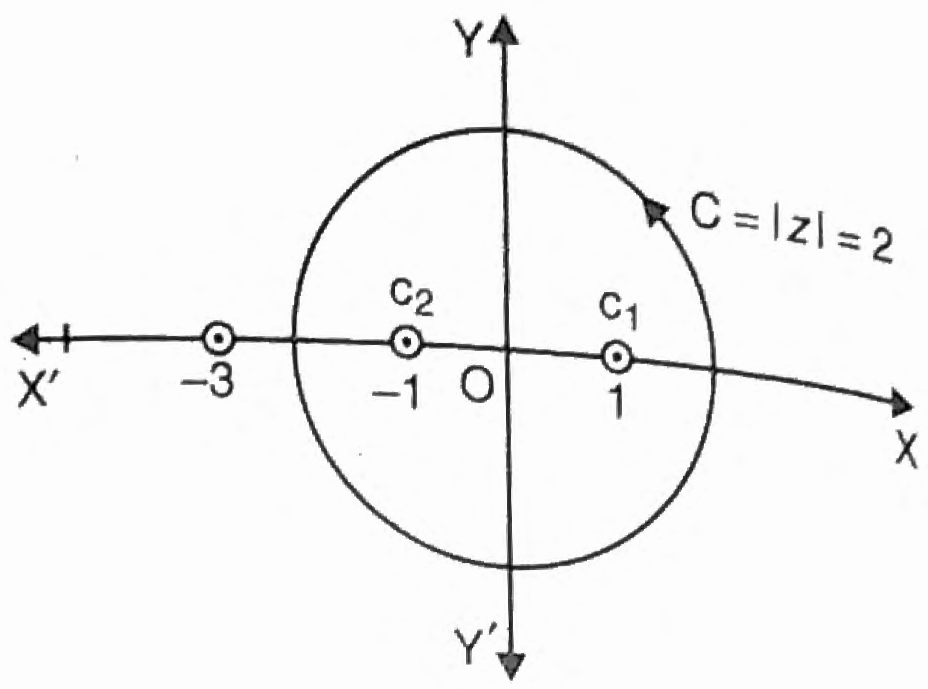
Solution. Poles of the integrand are given by putting the denominator equal to zero.

i.e., $(z^2-1)(z+3) = 0$

$\Rightarrow$ $z = 1, -1, -3$ (Simple poles)

The circle $|z| = 2$ has centre at $z = 0$ and radius 2.

Clearly the poles $z = 1$ and $z = -1$ lie inside the given circle while the pole $z = -3$ lies outside it.



$$\therefore \int_C \frac{3z^2+z+1}{(z^2-1)(z+3)}\,dz$$

$$= \int_{C_1} \frac{\frac{3z^2+z+1}{(z+1)(z+3)}}{z-1}\,dz + \int_{C_2} \frac{\frac{3z^2+z+1}{(z-1)(z+3)}}{z+1}\,dz$$

$$= 2\pi i\left[\frac{3z^2+z+1}{(z+1)(z+3)}\right]_{z=1} + 2\pi i\left[\frac{3z^2+z+1}{(z-1)(z+3)}\right]_{z=-1}$$ (Using Cauchy's Integral formula)

$$= 2\pi i\left(\frac{5}{8}\right) + 2\pi i\left(-\frac{3}{4}\right)$$

$$= 2\pi i\left(\frac{-1}{8}\right) = -\frac{\pi i}{4}$$

Which is the required value of the given integral. **Ans.**

Example 45. *Use Cauchy's integral formula to evaluate* $\int_C \frac{e^{3z}}{(z+1)^4}\,dz$ *where C is the circle* $|z| = 2$.

Solution. The integrand has singularity $z = -1$ which lies inside the given circle.

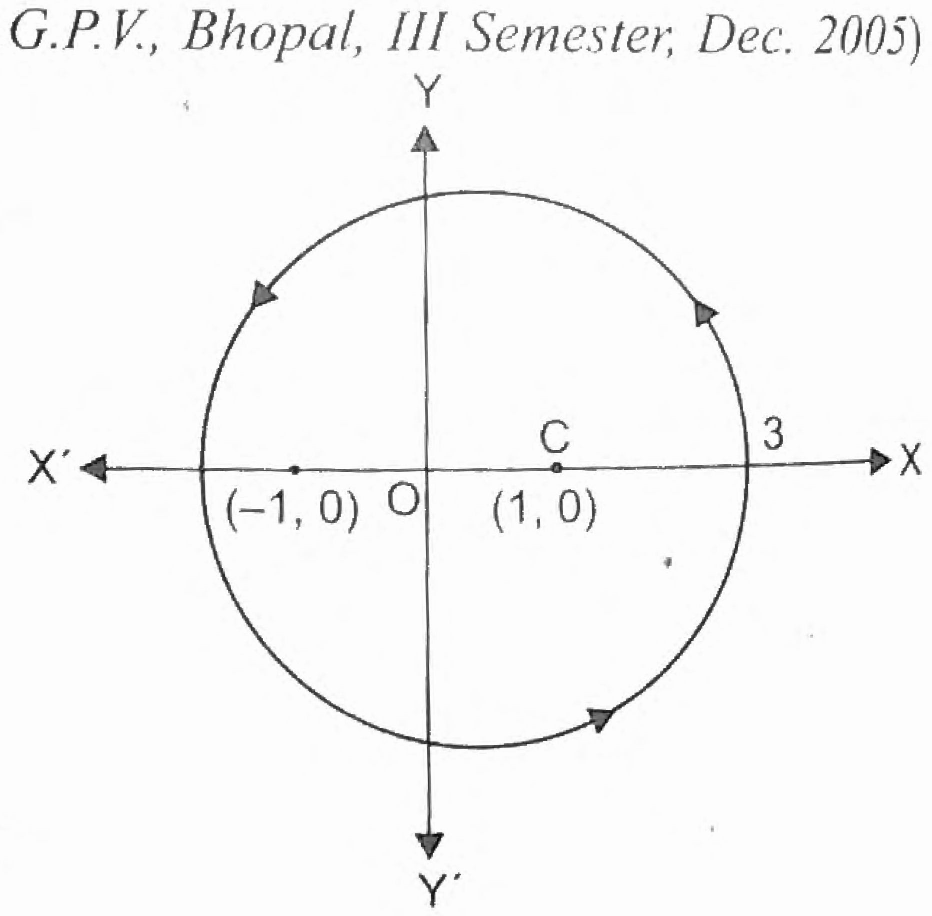
$$\int_C \frac{e^{3z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!}\left[\frac{d^3}{dz^3}(e^{3z})\right]_{z=-1} = \frac{\pi i}{3}\left(27e^{3z}\right)_{z=-1} = \frac{9\pi i}{e^3}.$$ **Ans.**

Example 46. Evaluate $\oint_C \frac{e^z}{(z+1)^2}\,dz$, *where C is the circle* $|z-1| = 3$.

(*R.G.P.V., Bhopal, III Semester, Dec. 2005*)

Solution. We have,

$\oint_C \frac{e^z}{(z+1)^2}$, where C is the circle with centre (1, 0) and radius 3.



The pole of the integral are determined by putting the denominator equal to zero.

$\Rightarrow$ $(z+1)^2 = 0$

$z = -1, -1$

Here there is a pole at $z = -1$ of order 2.

By Cauchy Integral Theorem for the derivative of a function

$$\left[\oint_C \frac{f(z)}{(z-a)^n} = \frac{2\pi i}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}f(a)\right]$$

$$\oint_C \frac{e^z}{(z+1)^2}\,dz = \frac{2\pi i}{1!}\left[\frac{d}{dz}e^z\right]_{z=-1}$$

$$= 2\pi i[e^z]_{z=-1}$$

$$= 2\pi i\, e^{-1} = \frac{2\pi i}{e}$$

Which is the required value of the given integral. **Ans.**

Example 47. *Using Cauchy's integral formula, evaluate* $\frac{1}{2\pi i}\int_C \frac{ze^z}{(z-a)^3}\,dz$, *where the point a lies within the closed curve C.*

Solution. $\int_C \frac{ze^z}{(z-a)^3}\,dz$

$$= \frac{2\pi i}{2!}\left[\frac{d^2}{dz^2}(ze^z)\right]_{z=a} = \frac{2\pi i}{2}\left[\frac{d}{dz}\{(z+1)e^z\}\right]_{\text{at } z=a}$$

$$= \frac{2\pi i}{2}\left[(z+1)e^z + e^z\cdot 1\right]_{\text{at } z=a} = \frac{2\pi i}{2}\left[(z+2)e^z\right]_{\text{at } z=a}$$

$$= 2\pi i\,\frac{(a+2)e^a}{2}$$

$$= \pi i(a+2)e^a$$

$$\frac{1}{2\pi i}\int_C \frac{ze^z}{(z-a)^3}\,dz = \frac{\pi i}{2\pi i}(a+2)e^a$$

$$= \frac{1}{2}(a+2)e^a$$

Which is the required value of the given integral. **Ans.**

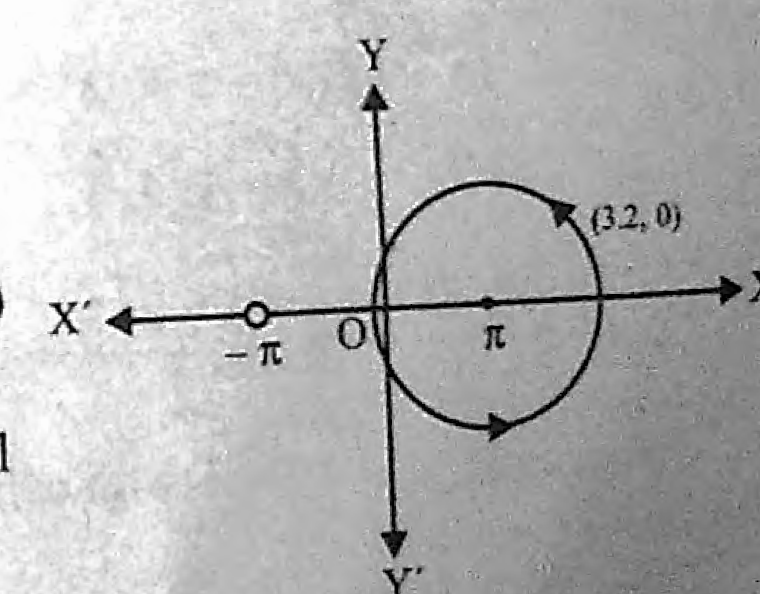
Example 48. *Derive Cauchy Integral Formula.*

Evaluate $\int_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$.

where C is the circle $|z-\pi| = 3.2$

Solution. Here,

$$I = \int_C \frac{e^{3iz}}{(z+\pi)^3}\,dz$$



Where C is a circle $\{|z-\pi| = 3.2\}$ with centre $(\pi, 0)$ and radius 3.2.

Poles are determined by putting the denominator equal to zero.

$(z+\pi)^3 = 0 \Rightarrow z = -\pi, -\pi, -\pi$

There is a pole at $z = -\pi$ of order 3.

But there is no pole within C.

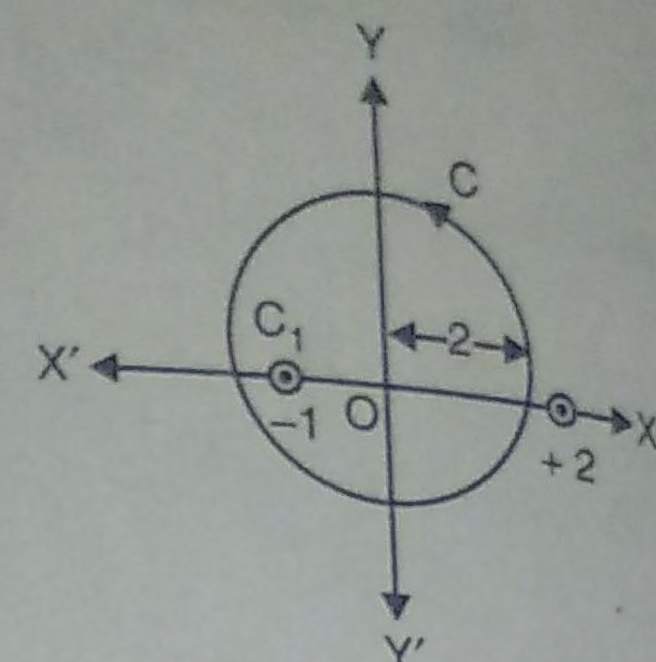
By Cauchy Integral Formula.

$$\int_C \frac{e^{3z}}{(z+\pi)^3}\,dz = 0 \qquad \textbf{Ans.}$$

Example 49. *Evaluate* $\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ *where* C *is* $|z-i| = 2$.

Solution. The centre of the circle is at $z = i$ and its radius is 2. Poles are obtained by putting the denominator equal to zero. i.e.; $(z+1)^2(z-2) = 0 \Rightarrow z = -1, -1, 2$.

The integrand has two poles at $z = -1$ (second order) and $z = 2$ (simple pole) of which $z = -1$ is inside the given circle.



$$\int_C \frac{(z-1)dz}{(z+1)^2(z-2)} = \int_{C_1} \frac{\frac{z-1}{z-2}}{(z+1)^2}\,dz$$

[By Cauchy's Integral formula $\int \frac{f(z)}{(z+1)^2}\,dz = 2\pi i f'(-1)$

Here, $$f(z) = \frac{z-1}{z-2}$$

$\Rightarrow$ $$f'(z) = \frac{(z-2).1-(z-1).1}{(z-2)^2} = \frac{-1}{(z-2)^2},$$

$\Rightarrow$ $$f'(-1) = \frac{-1}{(-1-2)^2} = \frac{-1}{9}$$

$\therefore$ $$\int \frac{(z-1)}{(z+1)^2(z-2)}\,dz = 2\pi i\left(-\frac{1}{9}\right) = -\frac{2\pi i}{9} \qquad \textbf{Ans.}$$

Example 50. *Integrate* $\frac{1}{(z^3-1)^3}$ *the counter clock-wise sense around the circle* $|z-1| = 1$.

Solution. Poles of the given function are found by putting denominator equal to zero.

$$(z^3-1)^3 = 0$$

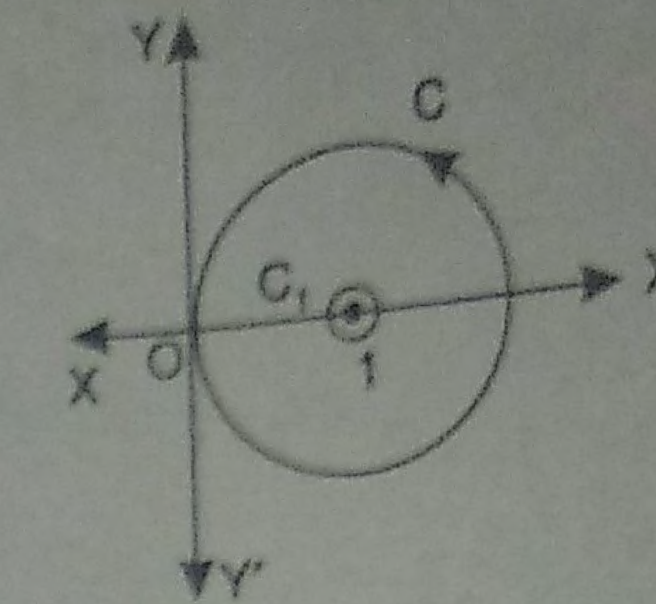
$$(z-1)^3(z^2+z+1)^3 = 0$$

$$z = 1, 1, 1,$$

$$z = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i,$$

The circle $|z-1| = 1$ with centre at $z = 1$ and unit radius encloses a pole of order three at $z = 1$.

By Cauchy Integral formula



$$\int_C \frac{1}{(z^3-1)^3}\,dz = \int_{C_1} \frac{1}{(z-1)^3(z^2+z+1)^3}\,dz = \int_{C_1} \frac{\frac{1}{(z^2+z+1)^3}}{(z-1)^3}\,dz$$

$$= \frac{2\pi i}{2}\left[\frac{d^2}{dz^2}\frac{1}{(z^2+z+1)^3}\right]_{z=1} = \pi i\left[\frac{d}{dz}\frac{-3(2z+1)}{(z^2+z+1)^4}\right]_{z=1}$$

$$= \pi i\left[\frac{(z^2+z+1)^4(-6)+3(2z+1)\,4(z^2+z+1)^3(2z+1)}{(z^2+z+1)^8}\right]_{z=1}$$

$$= \pi i\left[\frac{(z^2+z+1)(-6)+12(2z+1)(2z+1)}{(z^2+z+1)^5}\right]_{z=1}$$

$$= \pi i\left[\frac{(1+1+1)(-6)+12(2+1)(2+1)}{(1+1+1)^5}\right]$$

$$= \pi i\left[\frac{-18+108}{243}\right] = \frac{90}{243}\pi i = \frac{10}{27}\pi i \qquad \textbf{Ans.}$$

EXERCISE 13.2

Evaluate the following

1. $\int_C \frac{1}{z-a}\,dz$, where c is a simple closed curve and the point $z = a$ is
   (*i*) outside c; (*ii*) inside c. **Ans.** (*i*) 0 (*ii*) $2\pi i$
2. $\int_c \frac{e^z}{z-1}\,dz$, where c is the circle $|z| = 2$. **Ans.** $2\pi ie$
3. $\int_C \frac{\cos \pi z}{z-1}\,dz$, where c is the circle $|z| = 3$. **Ans.** $-2\pi i$
4. $\int_C \frac{\cos \pi z^2}{(z-1)(z-2)}\,dz$, where c is the circle $|z| = 3$. **Ans.** $4\pi i$
5. $\int_C \frac{e^{-z}}{(z+2)^5}\,dz$, where c is the circle $|z| = 3$. **Ans.** $\frac{\pi i e^2}{12}$